Lipschitz Decompositions of the Complements of Bilaterally Flat Sets

Jared Krandel

Department of Mathematics
Stony Brook University

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Lipschitz graph domains

Definition

We call an open set $\Omega \subseteq \mathbb{R}^{d+1}$ an M-**Lipschitz graph domain** if, after possibly translating and dilating Ω, there exists a function $r : \mathbb{S}^d \to \mathbb{R}^+$ such that

$$\partial\Omega = \{r(\theta)\theta : \theta \in \mathbb{S}^d\}$$

where for all $\theta, \psi \in \mathbb{S}^d$, we have

$$|r(\theta) - r(\psi)| \leq M|\theta - \psi|,$$

$$\frac{1}{M+1} \leq r(\theta) \leq 1.$$

Jones's Lipschitz decomposition result in $\mathbb{R}^2$

Theorem (Jones)

There is a constant $M > 0$ such that for any simply connected domain $\Omega \subseteq \mathbb{R}^2$ with $\mathcal{H}^1(\partial\Omega) < \infty$, there exists a rectifiable curve Γ, ($\mathcal{H}^1(\Gamma) < \infty$) such that

$$\Omega \setminus \Gamma = \bigcup_{j=1}^{\infty} \Omega_j$$

where $\{\Omega_j\}_j$ is a collection of disjoint M-Lipschitz graph domains satisfying

$$\sum_{j=1}^{\infty} \mathcal{H}^1(\partial\Omega_j) \leq M\mathcal{H}^1(\partial\Omega)$$

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Question

For $\Omega \subseteq \mathbb{R}^d$, $d > 2$, can we find geometric sufficient conditions on $\partial\Omega$ for the existence of Jones-type Lipschitz decompositions?

Ideas of proof of Jones's result

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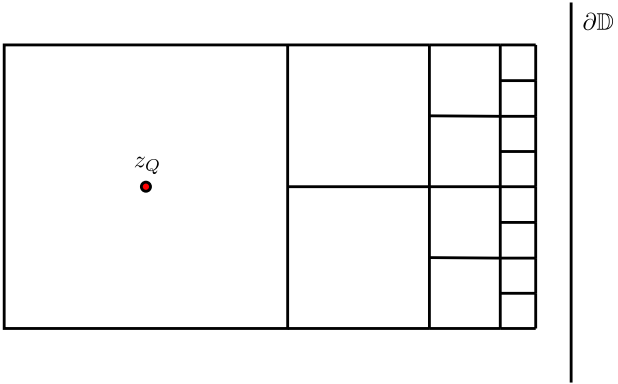

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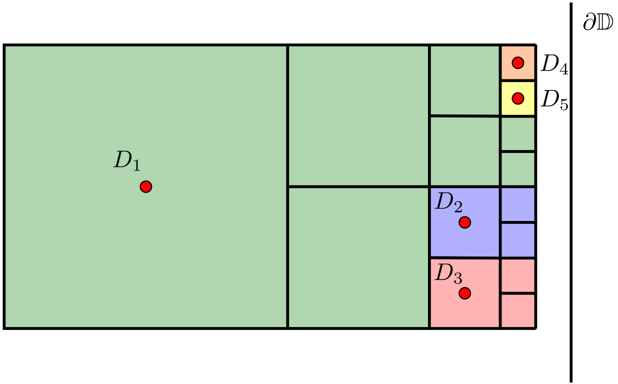

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(b) Riemann map $\to$ Reifenberg parameterization

(c) complex analysis estimates $\to$ Carleson packing estimates for a "corona decomposition" of $\partial\Omega$

(a) Reifenberg flat sets

Definition (Bilateral beta number)

For $E \subseteq \mathbb{R}^n$ and B a ball, the d-**bilateral beta number** for E inside B is

$$b\beta_E^d(B) = \frac{1}{\mathrm{diam}(B)} \inf_{P\ d\text{-plane}} d_H(B \cap E, B \cap P).$$

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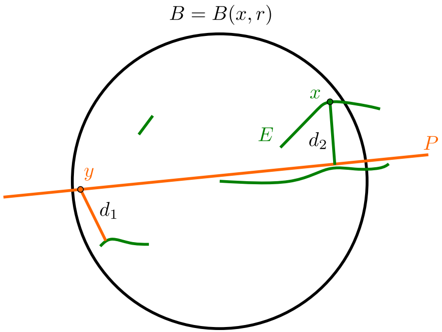

$$b\beta_E^d(B) = \inf_{P \ d-\text{plane}} \tfrac{1}{r} \max\{d_1, d_2\}$$

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We say that a set $E \subseteq \mathbb{R}^n$ is (ϵ, d)-**Reifenberg flat** if for all $x \in E$ and $r > 0$,

$$b\beta_E(B(x, r)) \leq \epsilon.$$

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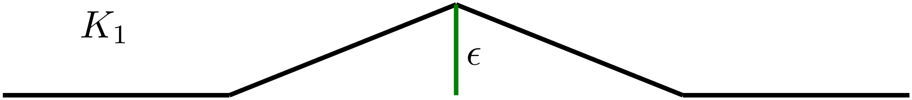

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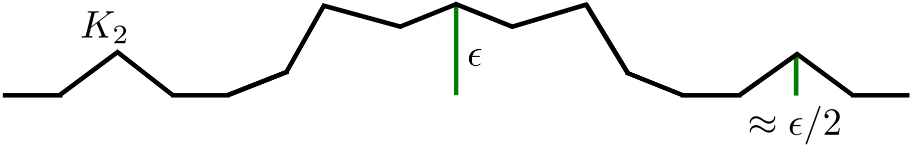

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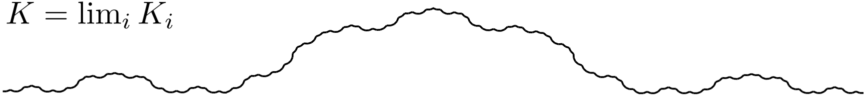
$K = \lim_i K_i$



(b) Reifenberg parameterization

Theorem (Reifenberg topological disk theorem)

If $E \subseteq \mathbb{R}^n$ is (ϵ, d)-Reifenberg flat with $0 \in E$ for small ϵ, then $\exists g : \mathbb{R}^n \to \mathbb{R}^n$ bi-Hölder which maps a d-disk onto $E \cap B(0,1)$. Specifically, g satisfies

$$C^{-1}|x-y|^{1+C\epsilon} \leq |g(x)-g(y)| \leq C|x-y|^{1-C\epsilon}$$

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$$E \cap B(0,1) = g(\mathbb{B}^d(0,1)).$$

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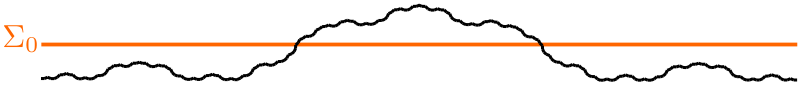
- G. David and T. Toro later gave an improved Reifenberg algorithm for "Reifenberg flat sets with holes" which also gave a condition for g to be bi-Lipschitz in terms of an upper bound on "accumulated wiggliness" over all scales at a given point.

(b) Reifenberg parameterization and the geometry of $\partial\Omega$

- Given Reifenberg flat E, the Reifenberg construction produces a collection $\{\Sigma_k\}_{k\geq 0}$ of smooth approximations to E on scale 2^{-k} used to construct g.



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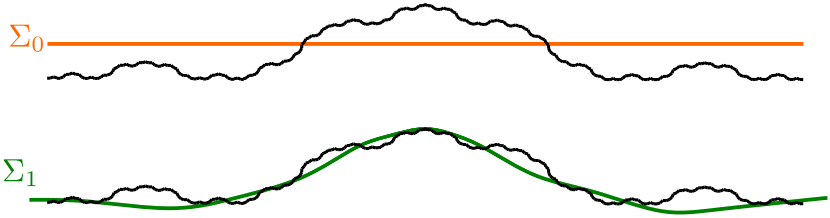

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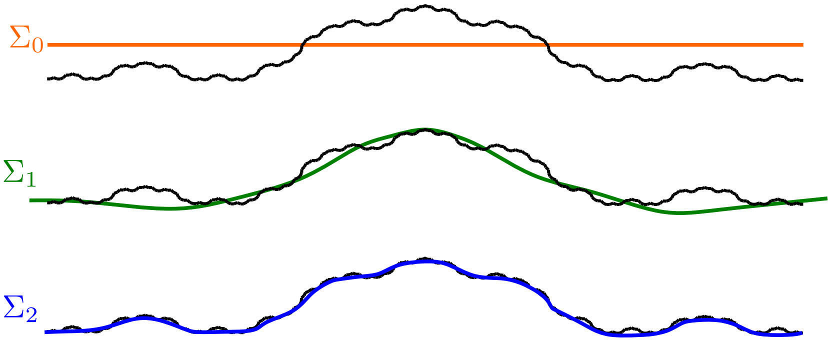

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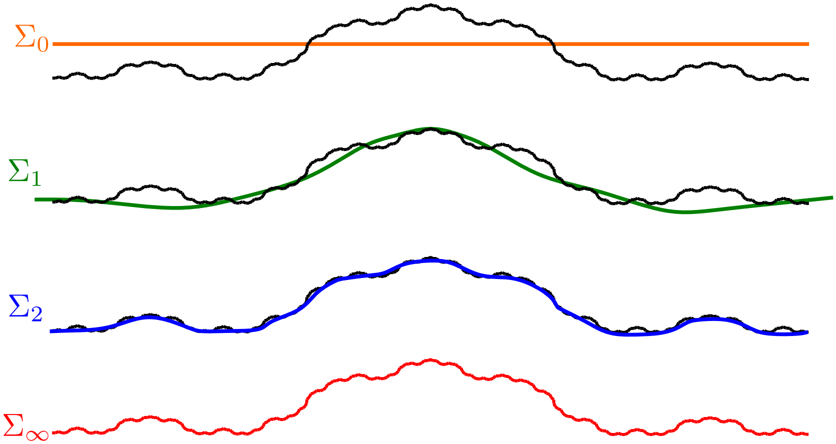

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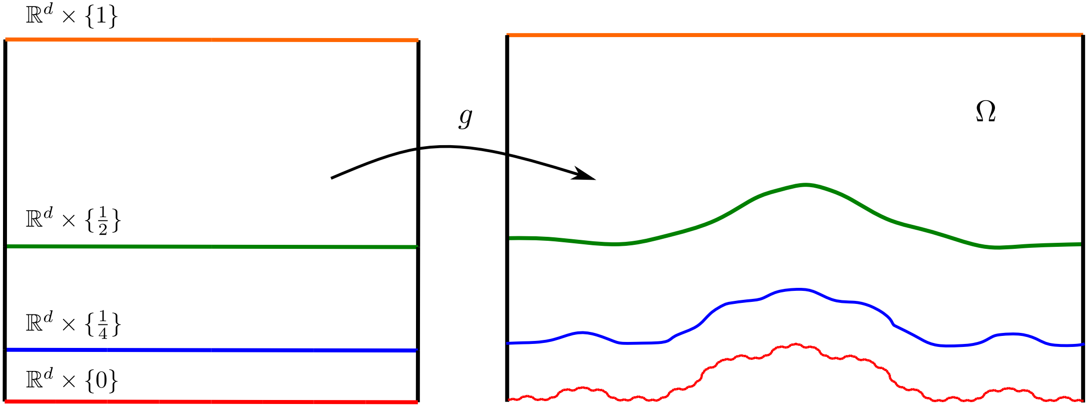
$g(x,t)$ parameterizes smooth Σ_t at "height" $\approx t$ above $\partial\Omega$ by interpolating between Σ_k's



(c) Reifenberg flat corona decompositions

Theorem (R. Schul, J. Azzam)

Let Σ *be* (ϵ, d)*-Reifenberg flat. Then* $\Delta(\Sigma \cap B(0,1)) = \bigcup_{S \in \mathscr{F}} S$ *where for each* S*, for every* $x \in Q \in S$*,*

(i) $\angle(P_{Q(S)}, P_Q) \leq \delta_0$ *(controlled tilting)*

(ii) $\sum_{x \in Q \in S} \beta_{cont}(Q)^2 \leq \epsilon_0$ *(controlled wiggliness)*

and

$$\sum_{S \in \mathscr{F}} \mathcal{H}^d(Q(S)) \lesssim_{\epsilon_0, \delta_0, d} \mathcal{H}^d(\Sigma \cap B(0,1)) \tag{1}$$

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Proposition (K.)

$\mathbb{H}^{d+1}$ *has a partition into Lipschitz graph domains* $\{D_j\}$ *such that* $Dg|_{D_j} \approx$ *constant, and the family* $\{\Omega_j\} = \{g(D_j)\}$ *is a collection of Lipschitz graph domains satisfying*

$$\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(0,1)) \lesssim_d \sum_{S \in \mathscr{F}} \mathcal{H}^d(Q(S)) \overset{(1)}{\lesssim}_{\epsilon_0, \delta_0, d} \mathcal{H}^d(\partial\Omega \cap B(0,1)).$$

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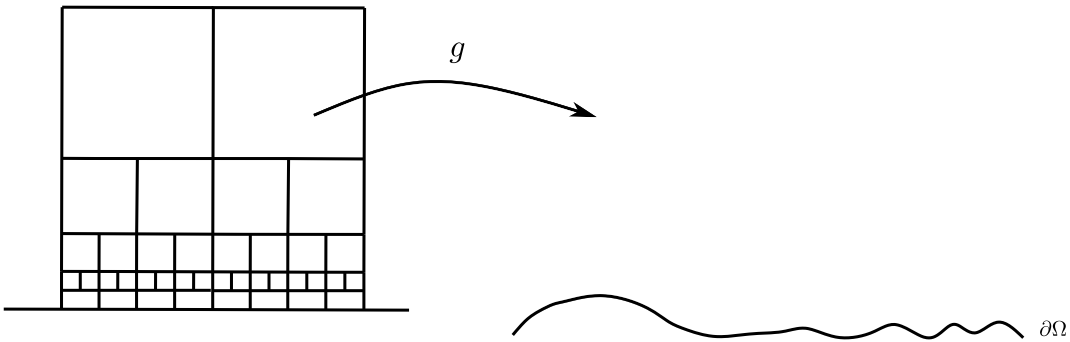
Proposition produces a corona decomposition!

- controlled tilting $\implies$ $g($"bottom" of $D_j)$ is a graph
- controlled wiggliness $\implies$ $g($"bottom" of $D_j)$ has Lipschitz constant $\lesssim \epsilon_0$

RF Corona decomposition $\iff$ disjoint Lipschitz graph domains



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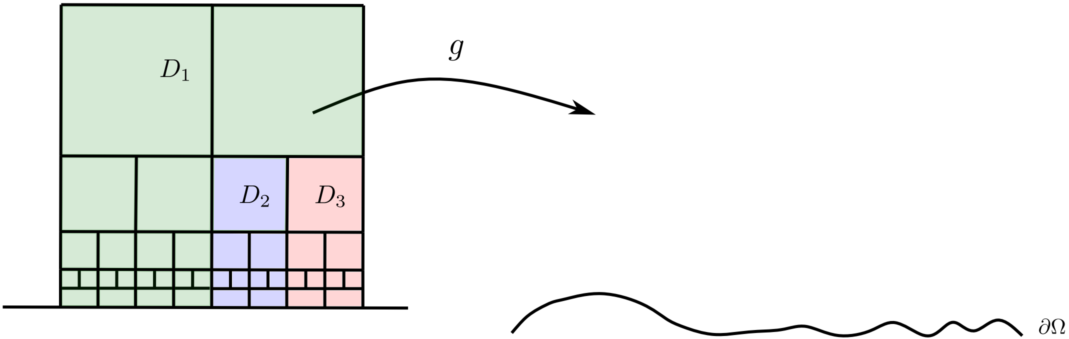

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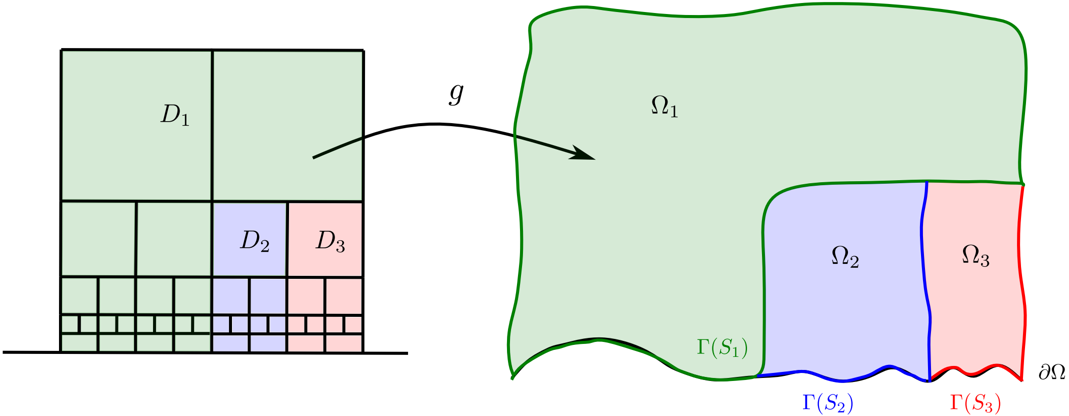
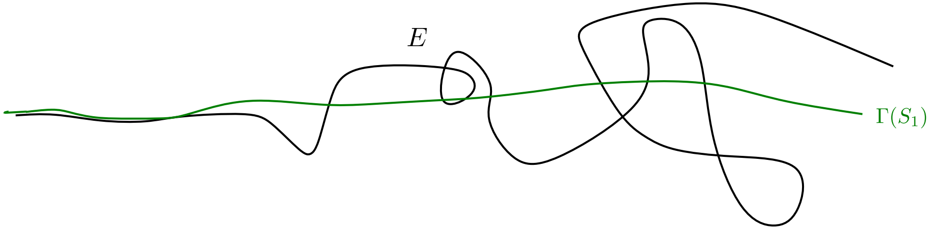


Corona decomposition $\Longleftrightarrow$ Lipschitz graph domains



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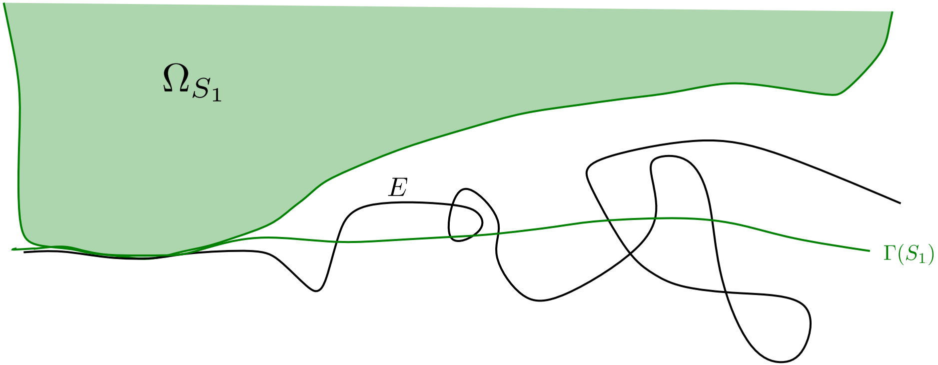

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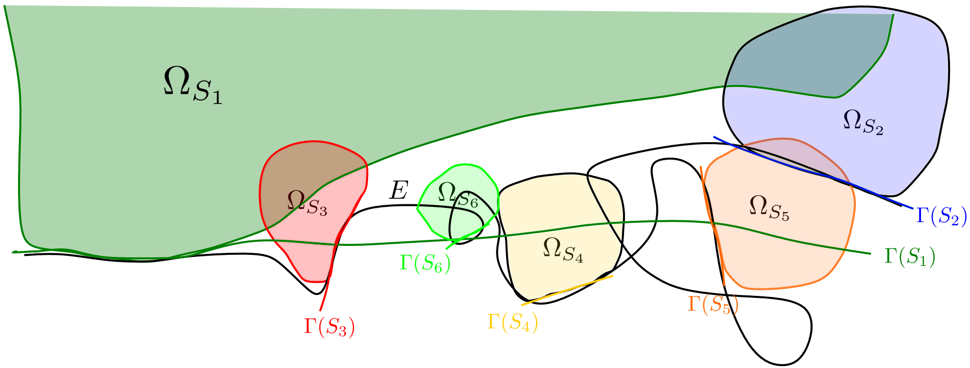

Jones's proof scheme with QGMT

To show that $\Omega \subseteq \mathbb{R}^{d+1}$ when $\partial\Omega$ is (ϵ, d)-Reifenberg flat+ has a local decomposition into Lipschitz graph domains...

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4. Prove $\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(0,1)) \leq M\mathcal{H}^d(\partial\Omega \cap B(0,1))$ by using Carleson packing estimates.

Reifenberg flat + Jones function bound

Theorem (K.)

For all $M > 0$, there is $\epsilon(d, M) > 0$ such that for any $\Omega \subseteq \mathbb{R}^{d+1}$, if

1. *$\partial\Omega$ is (ϵ, d) Reifenberg flat,*
2. *$\sum_{x \in Q} \beta_{cont}(Q)^2 \leq M$ for all $x \in \partial\Omega \cap B(0,1)$,*

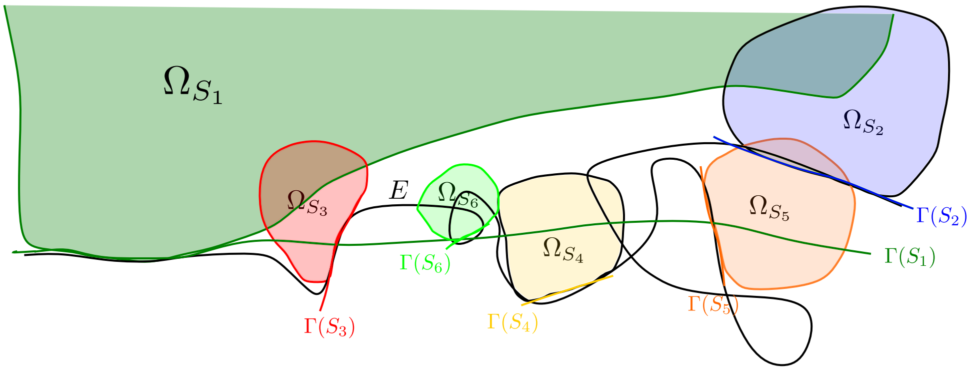
then there exists a d-rectifiable set Σ such that

$$\Omega \cap B(0,1) \setminus \Sigma = \bigcup_{j=1}^{\infty} \Omega_j$$

where Ω_j is M-Lipschitz and

$$\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(0,1)) \leq M\mathcal{H}^d(\partial\Omega \cap B(0,1))$$

No β^2 sum bound



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Theorem (K.)

There exist constants $M(d), \epsilon(d) > 0$ such that if $\partial\Omega$ is (ϵ, d)-Reifenberg flat, then there exists a collection of M-Lipschitz graph domains $\{\Omega_j\}$ such that

(a) $\Omega_j \subseteq \Omega$

(b) $\Omega \cap B(0,1) \subseteq \bigcup_{j=1}^{\infty} \Omega_j$

(c) $\exists C(d) > 0$ *such that* $\forall x \in \mathbb{R}^{d+1}$, $x \in \Omega_j$ *for at most* C j*'s*

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Theorem (K.)

Suppose $\partial\Omega$ is d-uniformly rectifiable. Then there exists $M > 0$ dependent on the uniform rectifiability constants such that there exists a collection of M-Lipschitz graph domains $\{\Omega_j\}$ such that (a), (b), (c) of the previous theorem hold, and

(d') $\sum_{j=1}^{\infty} \mathcal{H}^d(\partial\Omega_j \cap B(y,r)) \leq M\mathcal{H}^d(\partial\Omega \cap B(y,r))$ *for all* $y \in \partial\Omega \cap B(0,1)$ *and* $r > 0$.