Intro to Quantitative Geometric Measure Theory : All Scales and All Locations

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What is Geometric Measure Theory?

- Introductory real analysis: Study properties of real-valued functions. What does it mean for f to be...
  1. Continuous?
  2. Differentiable?
  3. Measurable?
  4. in L^p?

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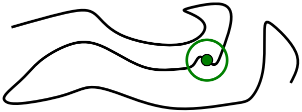
- Introductory real analysis: Study properties of real-valued functions. What does it mean for f to be...
  1. Continuous?
  2. Differentiable?
  3. Measurable?
  4. in L^p?
- Geometric measure theory: Use analysis to study geometric properties of sets and measures in $\mathbb{R}^n$.
  - How "flat" is a set? (How well is it approximated by d-planes?)

Tangent Lines



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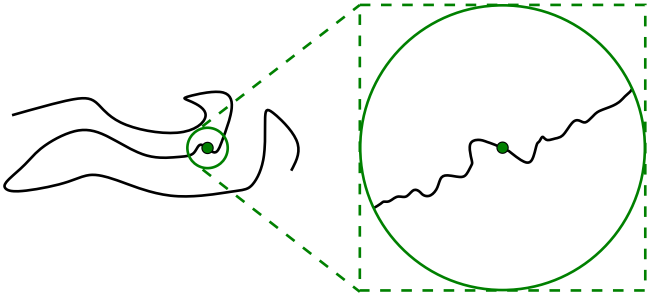

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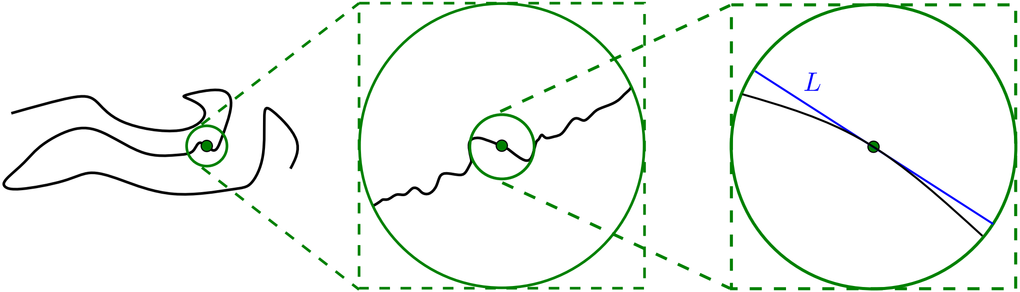

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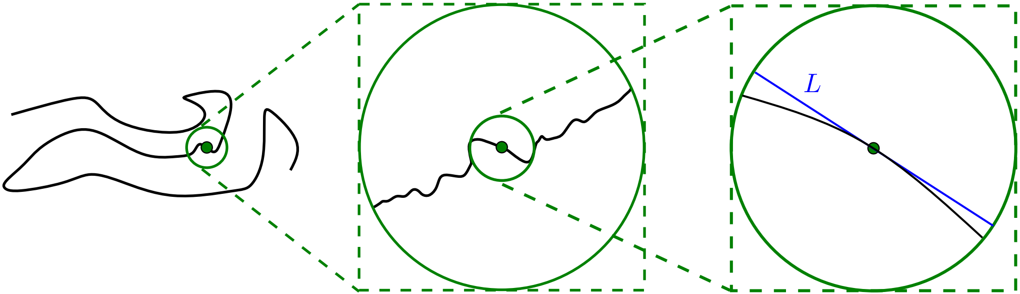

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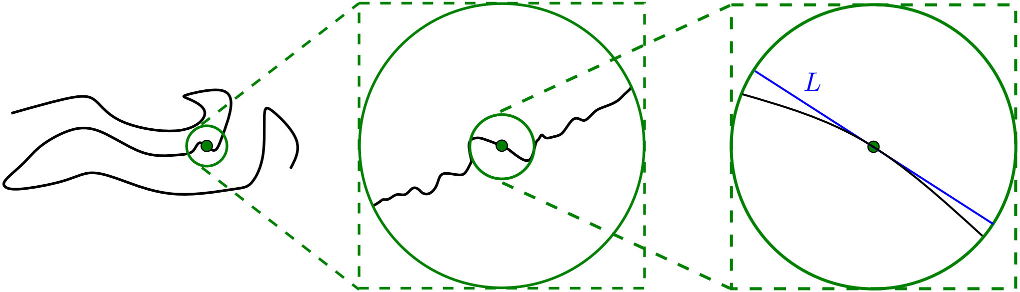

- Infinitesimal: Forgets about larger scales
- Not Quantitative: No bounds on degree of non-flatness

What is Quantitative Geometric Measure Theory?

Question

How do we **quantify** the degree to which a set is flat over **all scales and all locations**?

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How do we **quantify** the degree to which a set is flat over **all scales and all locations**?

Key construction: Good part and Bad part

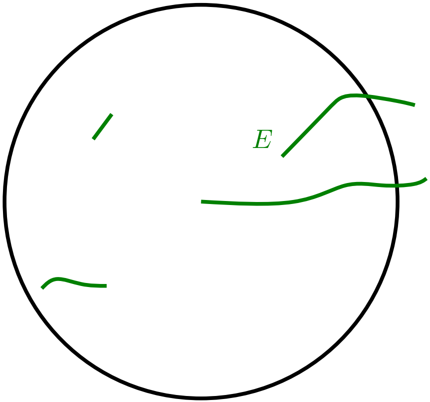
- Scale and location in $E \approx$ ball $B(x, t)$
- Classify scales and locations as good or bad:

$$\mathscr{G} \approx \{\text{balls where } E \text{ looks flat}\}$$

$$\mathscr{B} \approx \{\text{balls where } E \text{ does not look flat}\}$$

A Tool for Quantifying Flatness: Beta Number

$B = B(x, r)$



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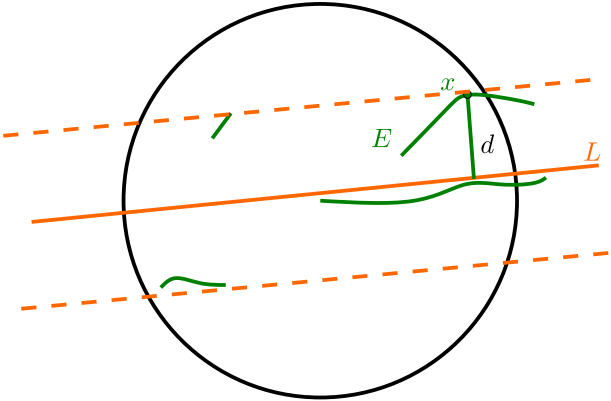

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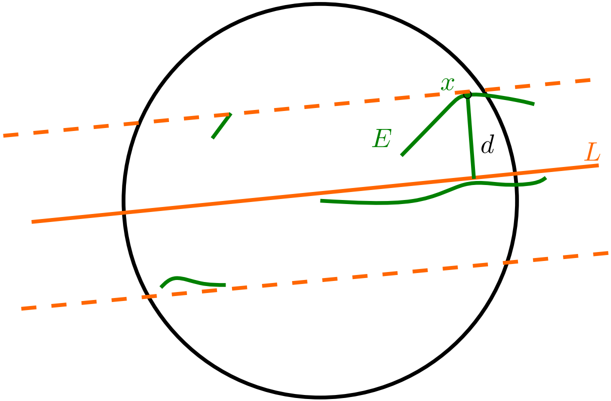

- $\beta_E(B) = \inf_L \frac{d}{r} = \frac{\text{width of thinnest strip containing } E \cap B}{r}$.
- $0 \leq \beta_E(B) \leq 1$.

A Tool for Isolating Scales and Locations: Cubes

Instead of balls, use "intrinsic" dyadic cubes $\Delta(E)$.

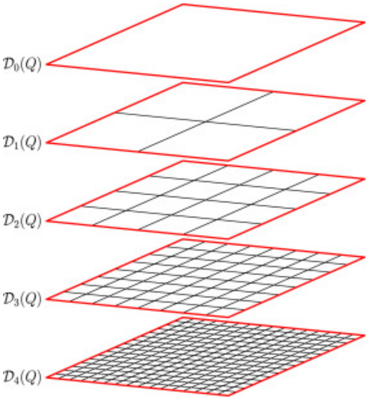

Figure: $\mathcal{D}(\mathbb{R}^n) = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$

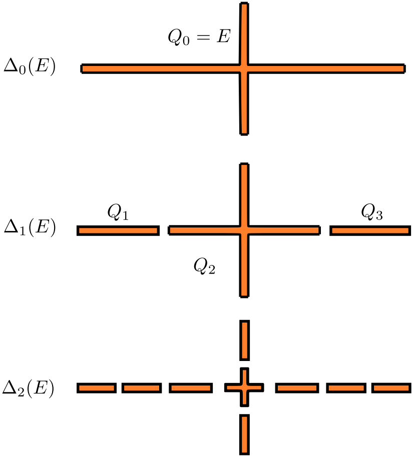

Figure: $\Delta(E) = \bigcup_{k \in \mathbb{Z}} \Delta_k(E)$

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Question

How do we **quantify** the degree to which a set is flat over **all scales and all locations**?

Key construction: Good part and Bad part

- Scale and location in $E \approx$ ball $B(x, t)$
- Classify scales and locations as good or bad. For example:

$$\mathscr{G} = \{Q \in \Delta(E) : \beta_E(Q) \leq \epsilon\}$$
$$\mathscr{B} = \{Q \in \Delta(E) : \beta_E(Q) > \epsilon\}$$

Regularity: Carleson Packing Condition

Question

How can we use $\mathscr{G}$ and $\mathscr{B}$ to quantitatively impose geometric regularity?

Regularity: Carleson Packing Condition

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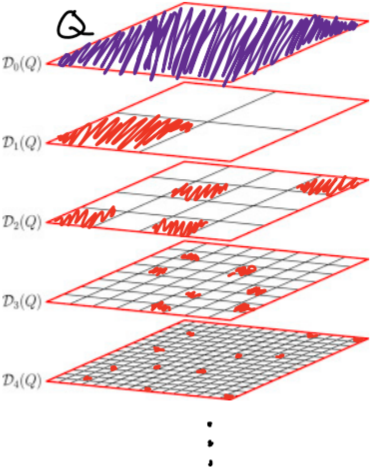
How can we use $\mathscr{G}$ and $\mathscr{B}$ to quantitatively impose geometric regularity?

Definition (Carleson Packing)

A family $\mathscr{B} \subseteq \Delta(E)$ satisfies a **Carleson packing condition** with constant C if, for all $Q \in \Delta(E)$,

$$\sum_{\substack{R \subseteq Q \\ R \in \mathscr{B}}} \mathrm{vol}(R) \leq C\mathrm{vol}(Q)$$

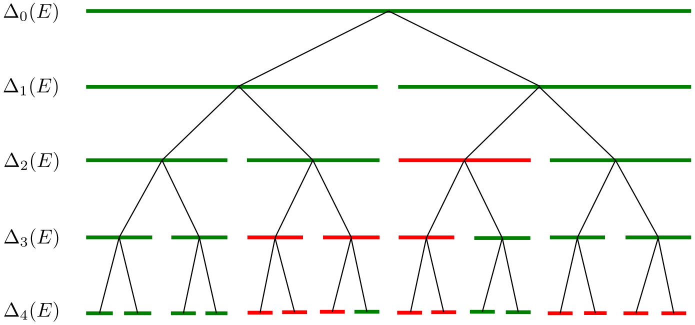
Regularity: Carleson Packing Condition



$$\sum_{\substack{R \subseteq Q \\ R \in \mathcal{B}}} \text{Area}(R) \leq C\, \text{Area}(Q)$$

Stopping Time Regions

We can “package” $\mathscr{G}$ into “connected” regions $\mathscr{F} = \{S_i\}_i$ for performing constructions.



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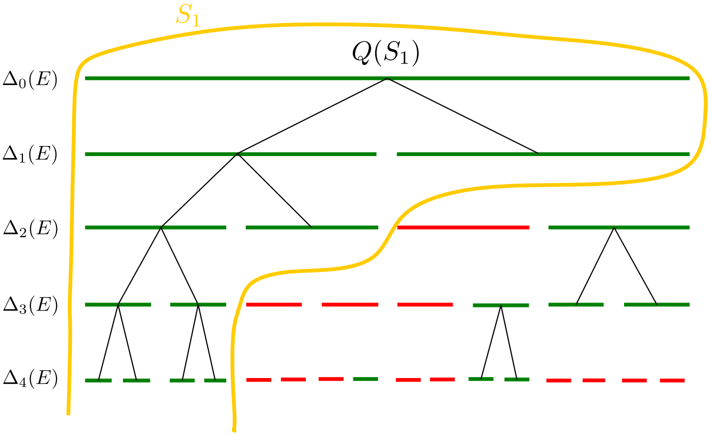

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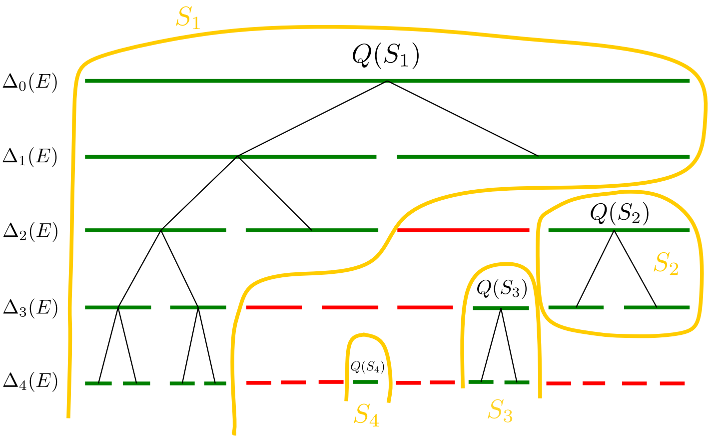

Corona Decomposition

Definition (Corona Decomposition)

$E \subseteq \mathbb{R}^n$ admits a **corona decomposition** if there exists a triple $(\mathscr{B}, \mathscr{G}, \mathscr{F})$ such that:

(i) $\mathscr{B}$ and $\{Q(S)\}_{S \in \mathscr{F}}$ satisfy Carleson packing conditions

(ii) For any $S \in \mathscr{F}$, there exists a 1-Lipschitz graph $\Gamma(S)$ such that for any $Q \in S$,

$$\mathrm{dist}(x, \Gamma(S)) \leq \mathrm{diam}(Q) \text{ for any } x \in 2Q.$$

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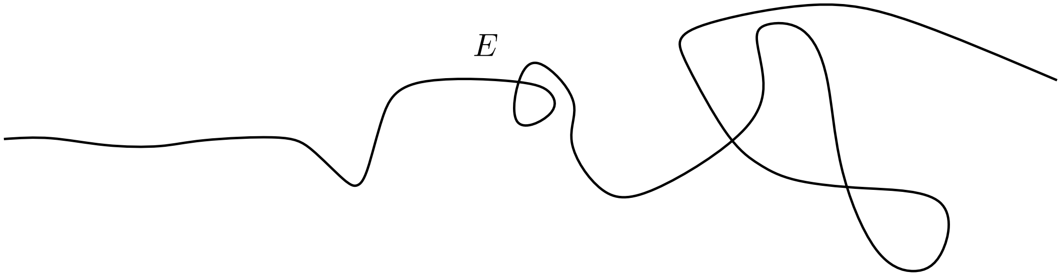

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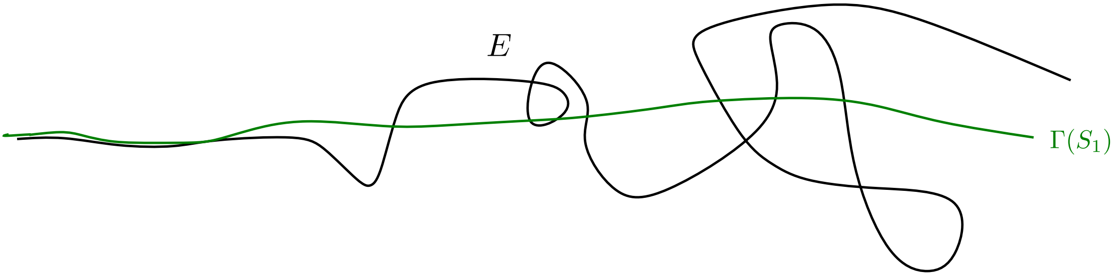

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