Topics in Quantitative Rectifiability: Densities, Lipschitz Decompositions, Big Pieces, and Traveling Salesmen

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Thesis Defense

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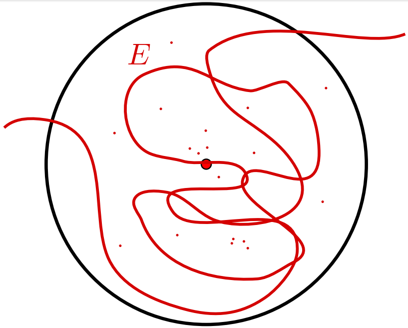
Rectifiability

Definition (n-rectifiable sets)

We say $E \subseteq X$ is n-rectifiable if there exists a countable collection of Lipschitz maps $f_i : A_i \subseteq \mathbb{R}^n \to X$ such that

$$\mathscr{H}^n\left(E \setminus \bigcup_i f_i(A_i)\right) = 0$$



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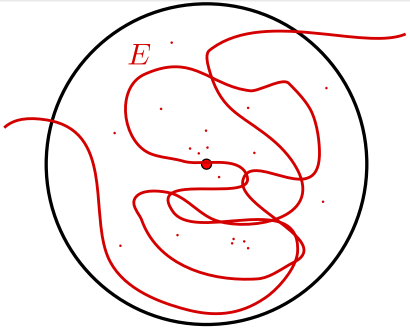

Theorem

Let $E \subseteq \mathbb{R}^d$ satisfy $\mathscr{H}^n(E) < \infty$. E is n-rectifiable if and only if E has an approximate tangent n-plane L_x at $\mathscr{H}^n$-a.e. $x \in E$. That is, for all $\epsilon > 0$,

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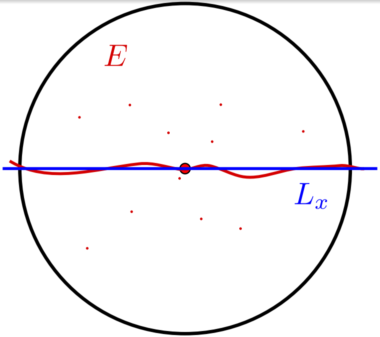

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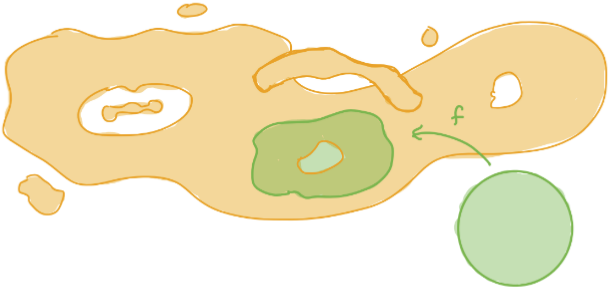
Definition (uniform n-rectifiability)

X is *uniformly n-rectifiable* if it is *Ahlfors n-regular*, i.e., there exists $C_0 > 0$ such that for all $x \in X$ and $0 < r < \mathrm{diam}(X)$,

$$C_0^{-1} r^n \leq \mathscr{H}^n(B(x,r)) \leq C_0 r^n,$$

and X has *Big Pieces of Lipschitz images of $\mathbb{R}^n$* (BPLI), i.e., there exist $L, \theta > 0$ such that for all $x \in X$ and $0 < r < \mathrm{diam}(X)$, there is an L-Lipschitz map $f : A \subseteq B^n(0,r) \to X$ such that

$$\mathscr{H}^n(B(x,r) \cap f(A)) \geq \theta r^n.$$



Qualitative vs Quantitative **Approximate Tangents**

n-Rectifiable $\iff$ "tangent plane" a.e. on infinitesimal scales

n-UR $\iff$ "coarse tangent plane" at "most" scales and locations

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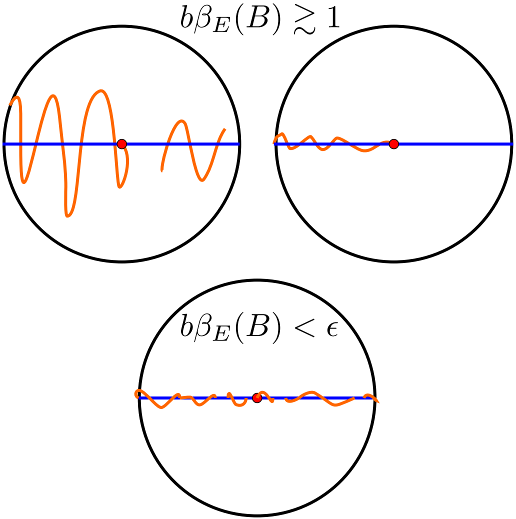

Definition (Bilateral Beta numbers)

Let $E \subseteq \mathbb{R}^d$ and for any ball B define

$$b\beta_E(B) = \inf_{P \text{ } n\text{-plane}} \frac{1}{\mathrm{diam}(B)} d_H(E \cap B, P \cap B)$$

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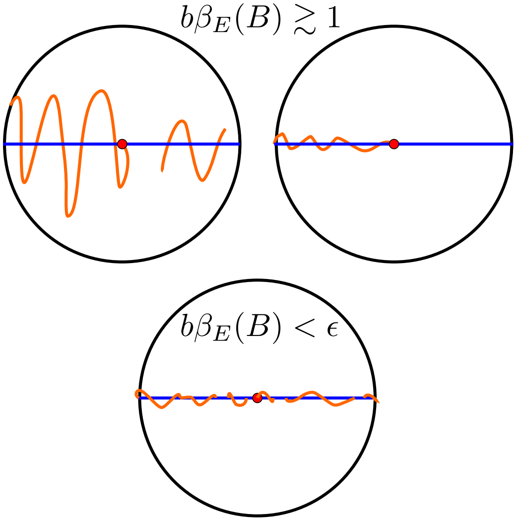

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Theorem (David and Semmes)

Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular. Then E is n-UR iff E satisfies the BWGL. That is, for all $\epsilon > 0$, the following set is a Carleson set:

$$\{(x,t) \in E \times (0, \operatorname{diam}(E)) \ : \ b\beta_E(x,t) > \epsilon\}$$

Our quantitative topics

1. Densities in uniformly rectifiable metric spaces: Quantitative regularity of Hausdorff measure,
2. Lipschitz decompositions: The existence of decompositions of domains with UR/RF boundary into a controlled number of nice pieces,
3. Stability of iterating the big pieces operator,
4. Quantitative rectifiability of curves: Relationships between the length of a curve and how non-flat it is at each scale and location,

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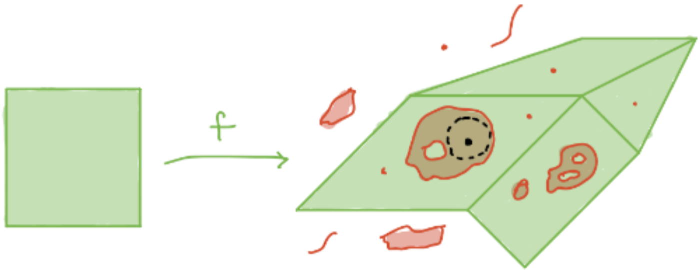
1. Densities

Theorem (Besicovitch, Mattila, Marstrand. Kircheim for $E \subseteq X$)

Let $E \subseteq \mathbb{R}^d$ be $\mathscr{H}^n$ measurable with $\mathscr{H}^n(E) < \infty$. E is n-rectifiable if and only if for $\mathscr{H}^n$-a.e. $x \in E$,

$$\lim_{r \to 0} \frac{\mathscr{H}^n(E \cap B(x,r))}{(2r)^n} = 1.$$

If $E \subseteq X$ is n-rectifiable, then the above equation holds.



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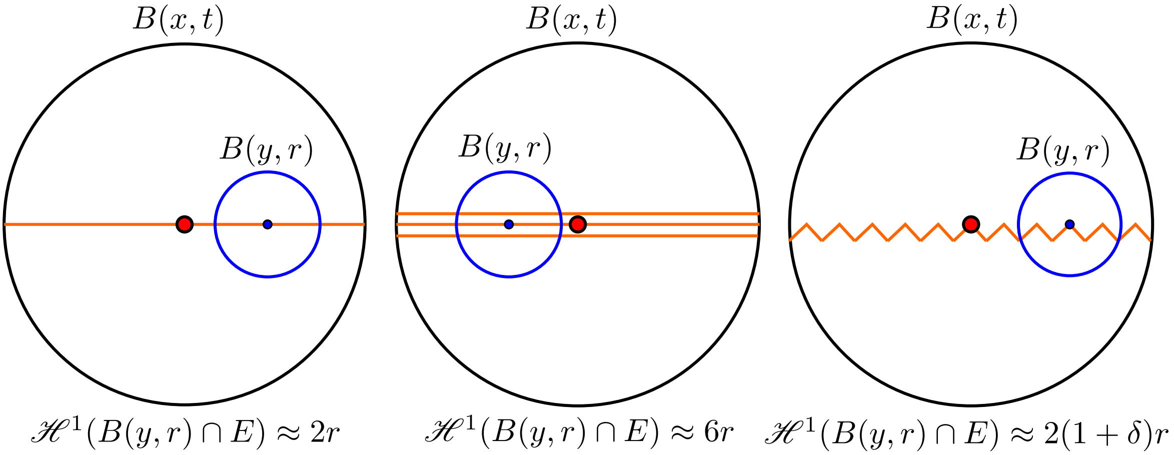

Definition (Weak constant density (WCD))

Let $E \subseteq \mathbb{R}^d, \epsilon > 0$ and define

$$\mathscr{G}(\epsilon) = \left\{(x,t) \in E \times \mathbb{R}^+ : \exists c_{x,t} > 0, \left|\frac{\mathscr{H}^n|_E(B(y,r))}{(2r)^n} - c_{x,t}\right| \leq \epsilon \text{ for } y \in B(x,t),\ r \gtrsim_\epsilon t\right\},$$

$$\mathscr{B}(\epsilon) = E \times \mathbb{R}^+ \setminus \mathscr{G}(\epsilon).$$

E satisfies the WCD if $\mathscr{B}(\epsilon)$ is a Carleson set for every $\epsilon > 0$.

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Theorem (K.)

Uniformly n-rectifiable metric spaces satisfy the WCD

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4. Quantitative rectifiability of curves: Relationships between the length of a curve and how non-flat it is at each scale and location,

2. Decompositions of domains

Definition (Whitney decomposition)

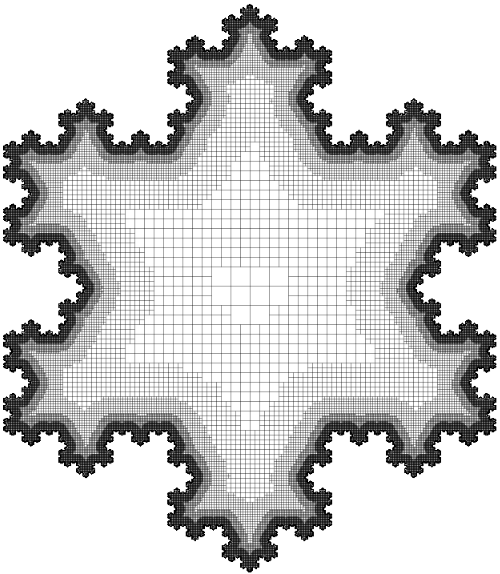
We say that $\mathcal{W}$ is a Whitney decomposition of a domain $\Omega \subseteq \mathbb{R}^{n+1}$ if $\mathcal{W}$ is a collection of closed cubes $\mathcal{W} = \{Q_j\}_{j \in \mathbb{N}}$ such that for all $Q \in \mathcal{W}$,

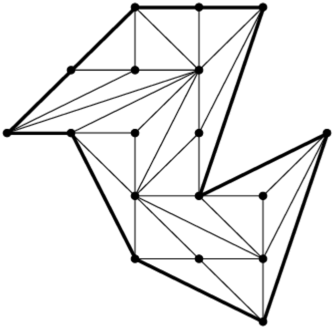
(i) $\Omega = \bigcup_{Q \in \mathcal{W}} Q$,

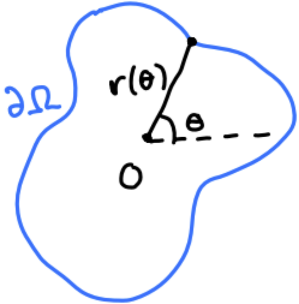
(ii) If $Q \neq Q'$, then Q and Q' have disjoint interiors,

(iii) $\mathrm{dist}(Q, \Omega^c) \asymp_n \mathrm{diam}(Q)$.

But $\displaystyle\sum_{Q \in \mathcal{W}} \mathscr{H}^n(\partial Q) = \infty$!







Definition (Lipschitz domains)

We say that a domain $\Omega \subseteq \mathbb{C}$ is an *M-Lipschitz domain* if, after a translation and dilation,

$$\partial\Omega = \left\{ r(\theta)e^{i\theta} : 0 \leq \theta \leq 2\pi \right\},$$

and for any $\theta, \psi \in [0, 2\pi]$,

$$|r(\theta) - r(\psi)| \leq M|\theta - \psi|$$

and

$$\frac{1}{1+M} \leq r(\theta) \leq 1$$

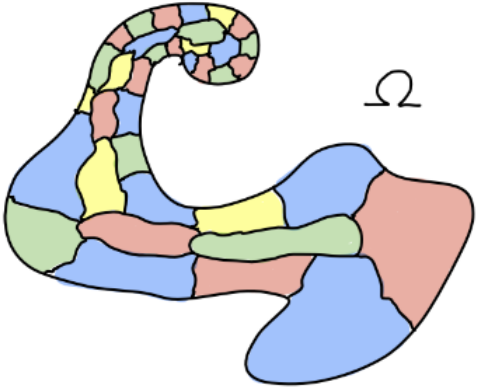
Theorem (Jones)

There is a constant $M > 0$ such that for any simply connected domain $\Omega \subseteq \mathbb{R}^2$ with $\mathscr{H}^1(\partial\Omega) < \infty$, there exists a finite length curve Γ, such that

$$\Omega \setminus \Gamma = \bigcup_{j=1}^{\infty} \Omega_j$$

where $\{\Omega_j\}_j$ is a collection of disjoint M-Lipschitz domains satisfying

$$\sum_{j=1}^{\infty} \mathscr{H}^1(\partial\Omega_j) \leq M \mathscr{H}^1(\partial\Omega)$$



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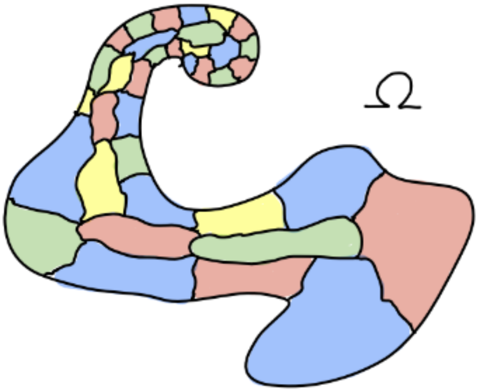

Question

What about higher dimensions? Let's assume $\partial\Omega$ is n-UR...

Theorem (K.)

There exist constants $M(n), A(n) > 0$ such that if $\Omega \subseteq \mathbb{R}^{n+1}$ is a domain where $\partial\Omega$ is n-UR, then there exists a collection of M-Lipschitz domains $\{\Omega_j\}$ such that

(i) $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$,

(ii) $\exists C(n) > 0$ *such that* $\forall x \in \mathbb{R}^{n+1}$, $x \in \Omega_j$ *for at most* C *values of* j,

(iii) *For any* $y \in \partial\Omega, \ 0 < r \leq 1$,

$$\sum_{j=1}^{\infty} \mathscr{H}^n(\partial\Omega_j \cap B(y,r)) \lesssim_{n,C_0,\theta,L} \mathscr{H}^n(B(y,Ar) \cap \partial\Omega).$$

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Theorem (K.)

There exists $\epsilon(n), A(n), M(n) > 0$ such that if $\partial\Omega$ is (ϵ, n)-Reifenberg flat then there exists a collection of M-Lipschitz domains $\{\Omega_j\}$ such that the conclusions of the previous theorem hold.

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1. Densities in uniformly rectifiable metric spaces: Quantitative regularity of Hausdorff measure,
2. Lipschitz decompositions: The existence of decompositions of domains with UR/RF boundary into a controlled number of nice pieces,
3. **Stability of the big pieces operator under iteration,**
4. Quantitative rectifiability of curves: Relationships between the length of a curve and how non-flat it is at each scale and location,

3. Big Pieces

Definition (Big pieces of $\mathscr{F}$)

Let $\mathscr{F}$ be a class of Ahlfors n-regular subsets of a metric space X and let $E \subseteq X$. We say that $E \in \mathrm{BP}(\mathscr{F})$ if E is Ahlfors n-regular and there exists $\theta > 0$ such that for all $x \in E$ and $0 < r < \mathrm{diam}(E)$, there exists $F_{x,r} \in \mathscr{F}$ such that

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Theorem (David and Semmes)

Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular. Then the following are equivalent

1. $E \in \mathrm{BP}(\mathrm{LI})$, *i.e., E is uniformly n-rectifiable,*
2. $E \in \mathrm{BP}^j(\mathrm{LI})$ *for* $j \geq 1$,
3. $E \in \mathrm{BP}^j(\mathrm{LG})$ *for* $j \geq 2$.

Theorem (K., Schul)

Let $\mathscr{F}$ be a class of Ahlfors n-regular sets in a metric space X. For any $j \geq 2$,

$$\mathrm{BP}^j(\mathscr{F}) \subseteq \mathrm{BP}^2(\mathscr{F}).$$

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Theorem (K., Schul)

Let $E \subseteq X$ be an Ahlfors n-regular set with $E \in \mathrm{BP}(\mathrm{BP}(\mathscr{F}))$. There exists a set $F \subseteq X$ such that

(i) $E \subseteq F$,

(ii) F *is Ahlfors n-regular.*

(iii) $F \in \mathrm{BP}(\mathscr{F})$.

The constants in the conclusion are quantitative with dependence on the constants in the assumptions.

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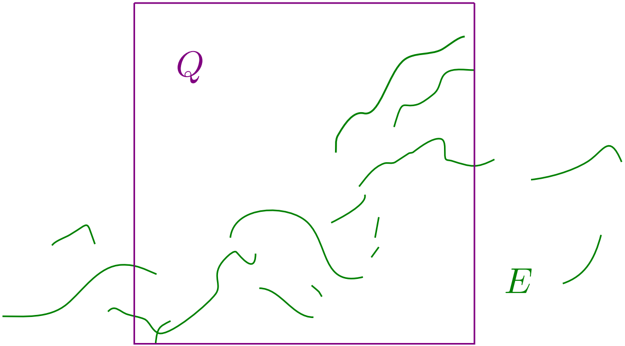

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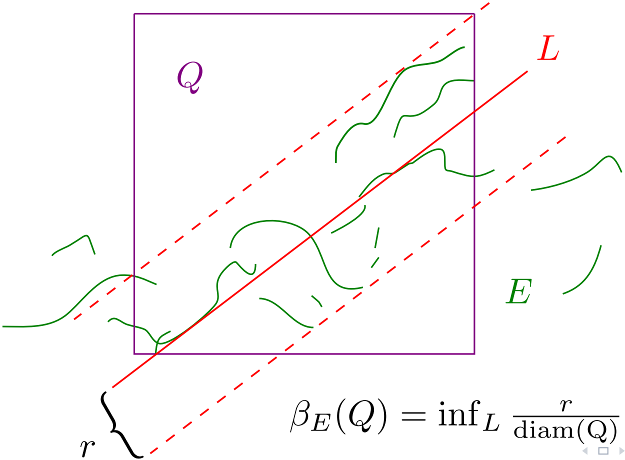

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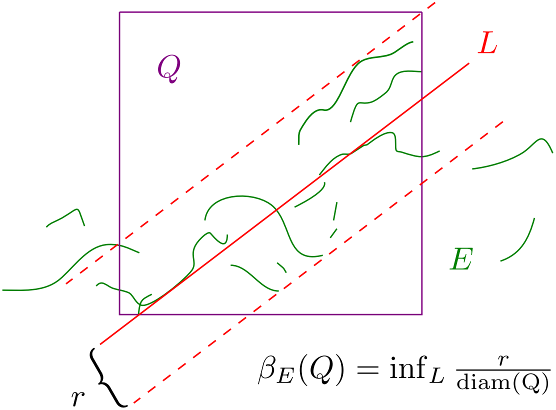

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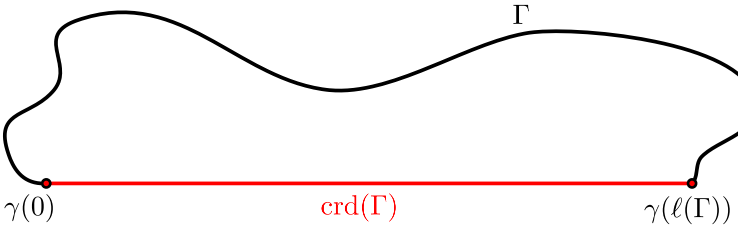
Theorem (Jones: $\mathbb{R}^2$, Okikiolu: $\mathbb{R}^n$, Schul: Hilbert space)

$E \subseteq \Gamma \subseteq \mathbb{R}^2$ with $\ell(\Gamma) < \infty$ if and only if

$$\mathrm{diam}(E) + \sum_{Q \in \mathcal{D}(\mathbb{R}^2)} \beta_E(3Q)^2 \,\mathrm{diam}(Q) < \infty.$$

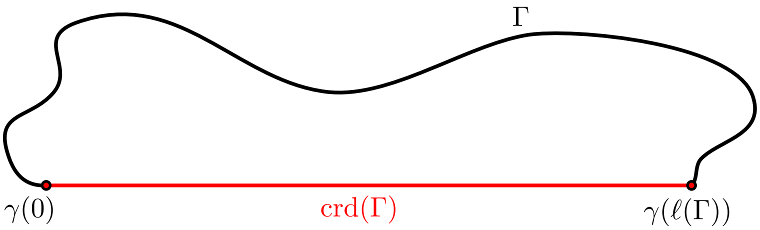
Moreover,

$$\ell(\Gamma) \asymp \mathrm{diam}(E) + \sum_{Q \in \mathcal{D}(\mathbb{R}^2)} \beta_E(3Q)^2 \,\mathrm{diam}(Q)$$



Theorem (Bishop)

Let $\Gamma \subseteq \mathbb{R}^d$ *be a Jordan arc. Then* $\sum_{Q\in\mathcal{D}(\mathbb{R}^d)} \beta_\Gamma(3Q)^2 \operatorname{diam}(Q) \asymp_d \ell(\Gamma) - \operatorname{crd}(\Gamma)$.



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Theorem (K.)

Let H *be a Hilbert space and let* $\Gamma \subseteq H$ *be a Jordan arc. For any multiresolution family* $\mathscr{H}$ *associated to* Γ *with inflation factor* $A > 200$, *we have*

$$\sum_{Q \in \mathscr{H}} \beta_\Gamma(Q)^2 \operatorname{diam}(Q) \asymp_A \ell(\Gamma) - \mathrm{crd}(\Gamma).$$

Thank You!

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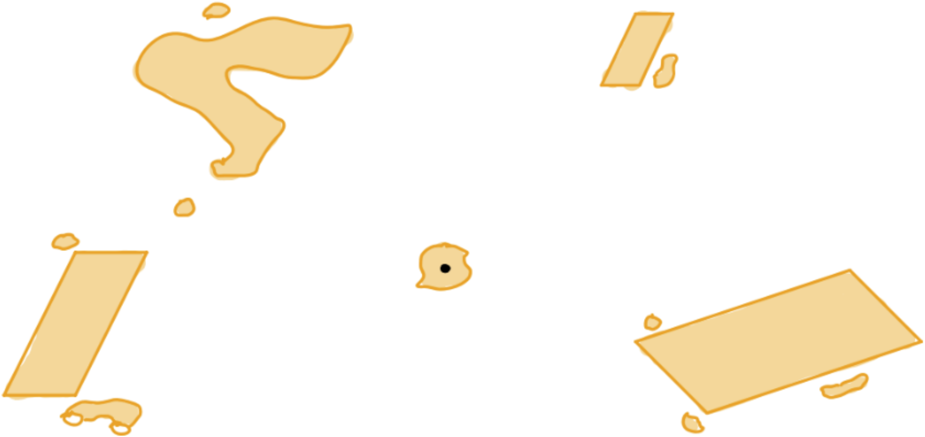
Theorem (David and Semmes in $\mathbb{R}^d$; Bate, Hyde, and Schul for metric spaces)

Let X be uniformly n-rectifiable. X has VBPBI, i.e., for all $\epsilon > 0$ there is an $L \geq 1$ such that for each $x \in X$ and $r > 0$ there exists $F \subseteq B(x, r)$, satisfying $\mathscr{H}^n_X(B(x, r) \setminus F) \leq \epsilon r^n$ and an L-bi-Lipschitz map $f : F \to \mathbb{R}^n$.



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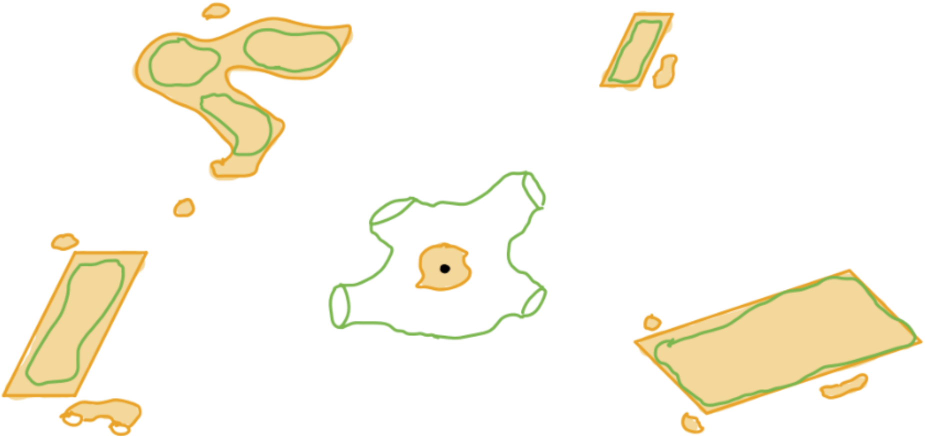

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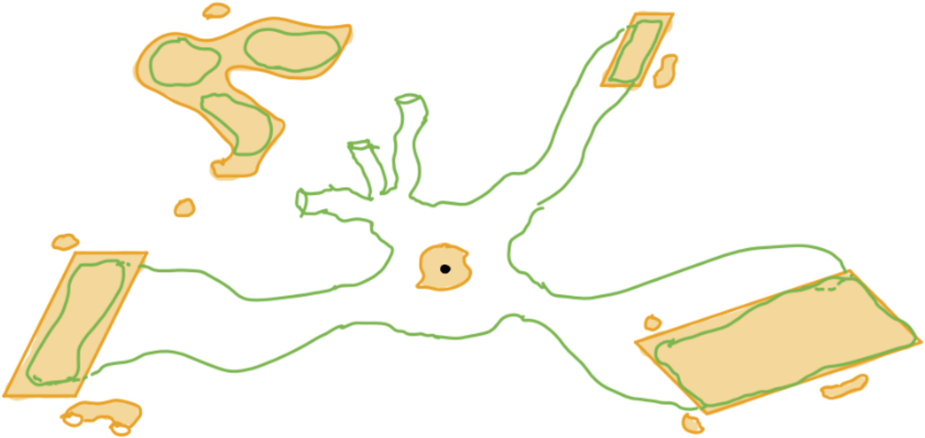

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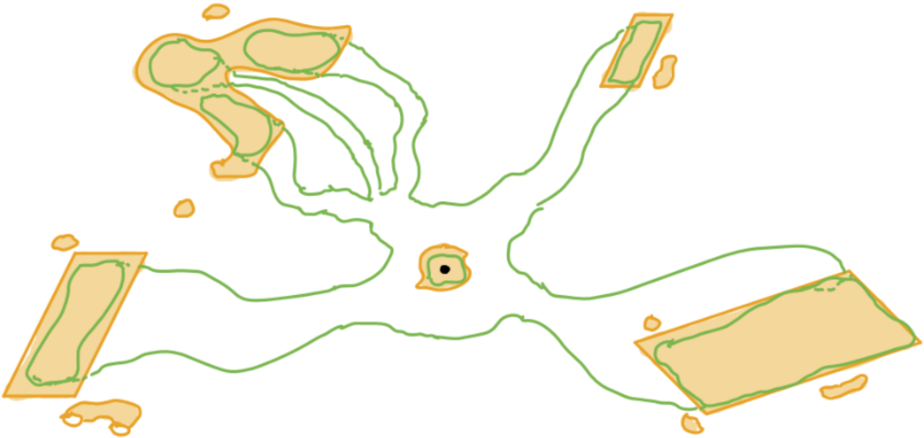
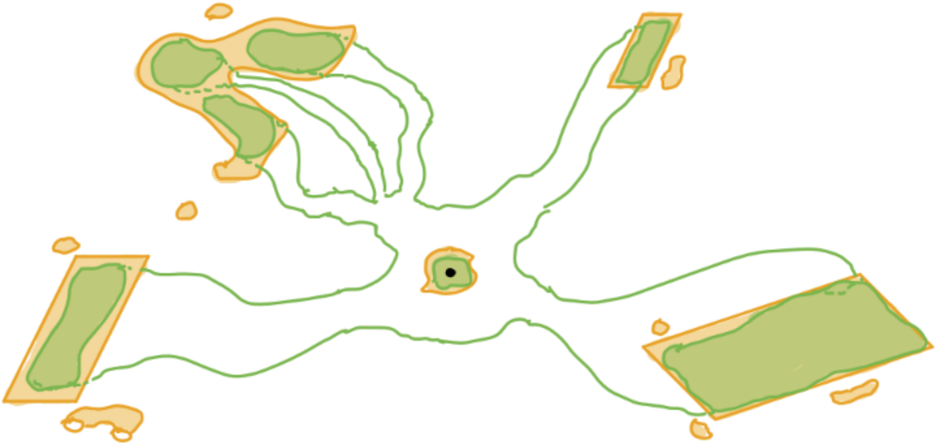


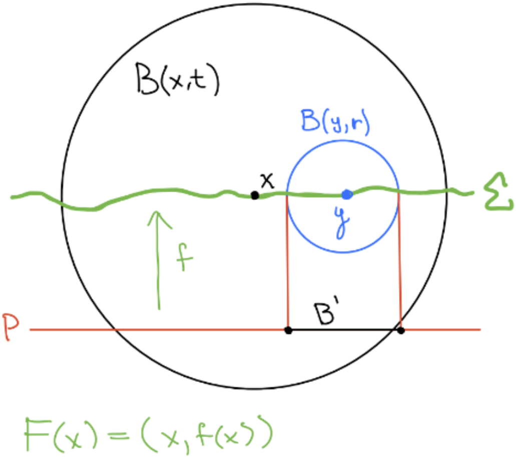
Theorem (David and Semmes in $\mathbb{R}^d$; Bate, Hyde, and Schul for metric spaces)

Let X be uniformly n-rectifiable. X has VBPBI, i.e., for all $\epsilon > 0$ there is an $L \geq 1$ such that for each $x \in X$ and $r > 0$ there exists $F \subseteq B(x, r)$, satisfying $\mathscr{H}^n_X(B(x, r) \setminus F) \leq \epsilon r^n$ and an L-bi-Lipschitz map $f : F \to \mathbb{R}^n$.



Let $f : [0,1]^n \twoheadrightarrow \Sigma$ be L-bi-Lipschitz.

1. By the area formula, $\mathscr{H}^n|_\Sigma$ can be calculated by integrating $\mathscr{J}_f = \text{Jac}(Df)$.



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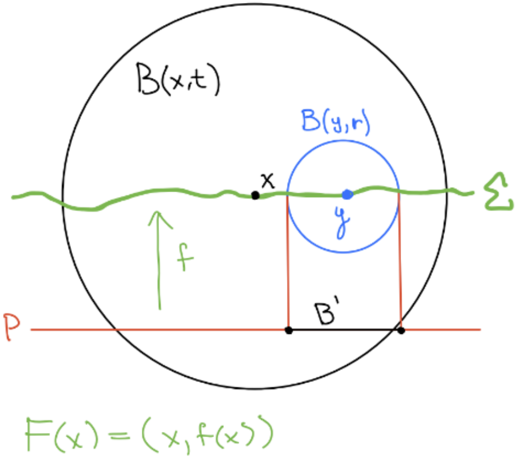

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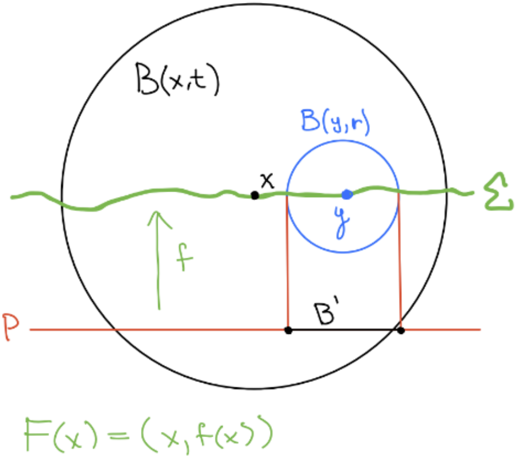
3. Changes in the averages of $\mathscr{J}_f \in L^2 \cap L^\infty$ mean large wavelet coefficients.



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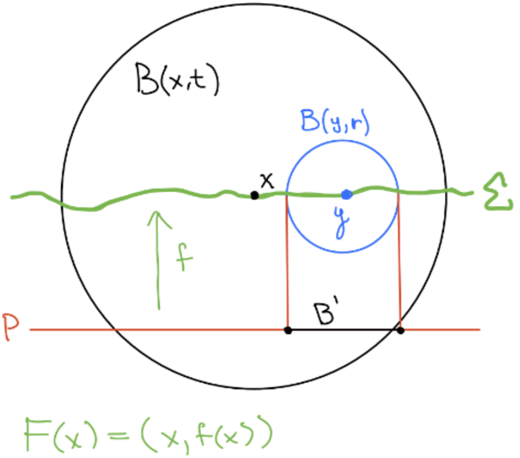
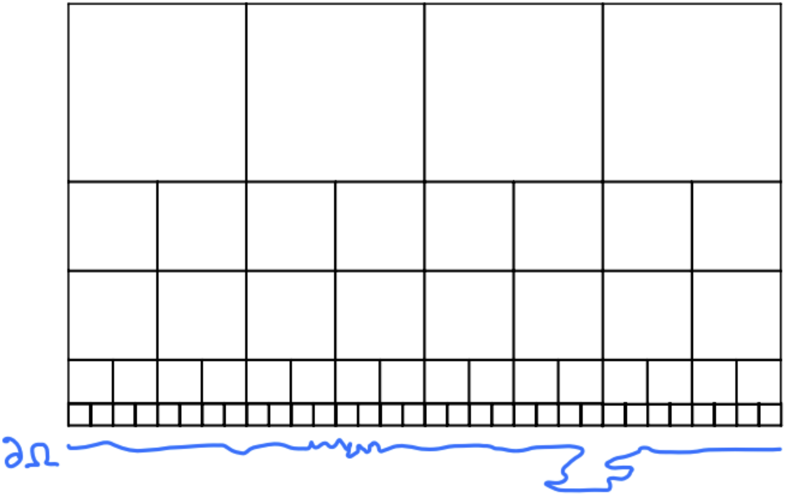
4. Therefore, densities cannot change too often.



1. Let $\mathcal{W}$ be a Whitney decomposition of Ω. Declare $Q \in \mathcal{W}$ **Good** if $b\beta_{\partial\Omega}(10Q) \leq \epsilon$.



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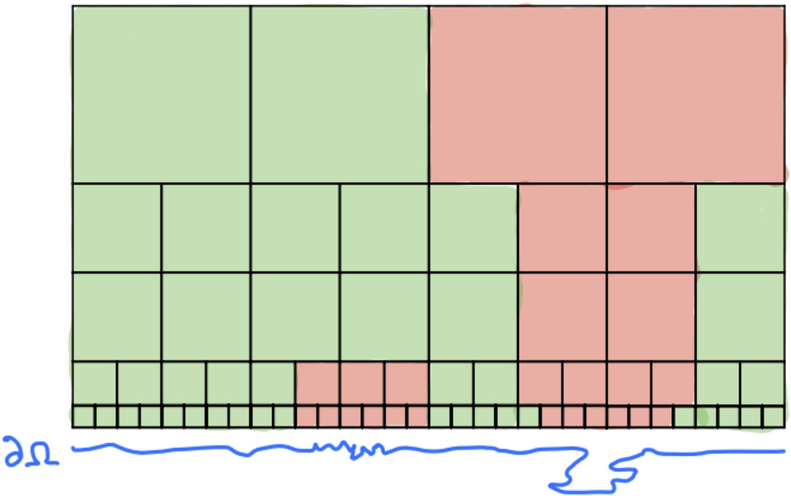

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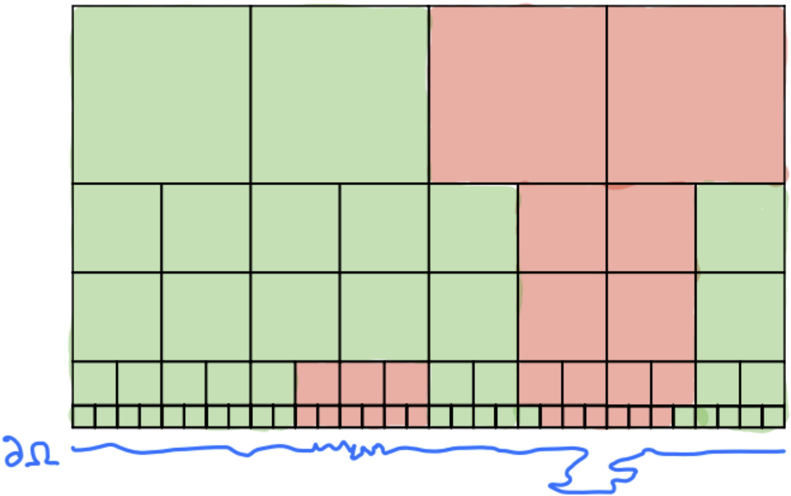

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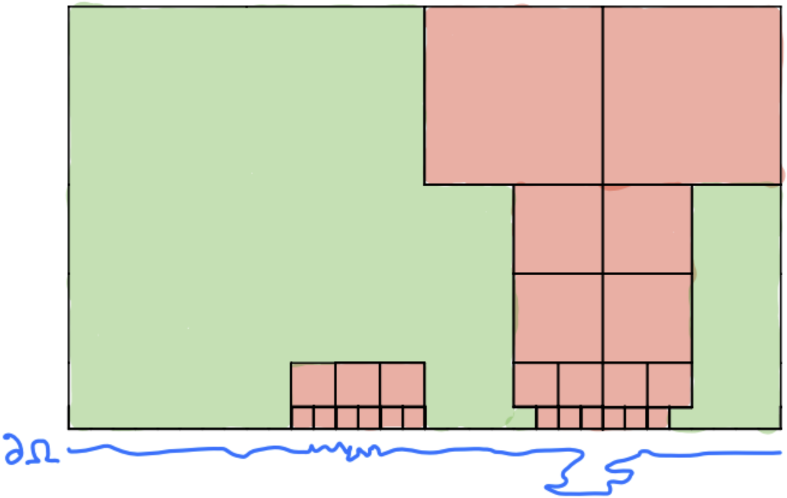

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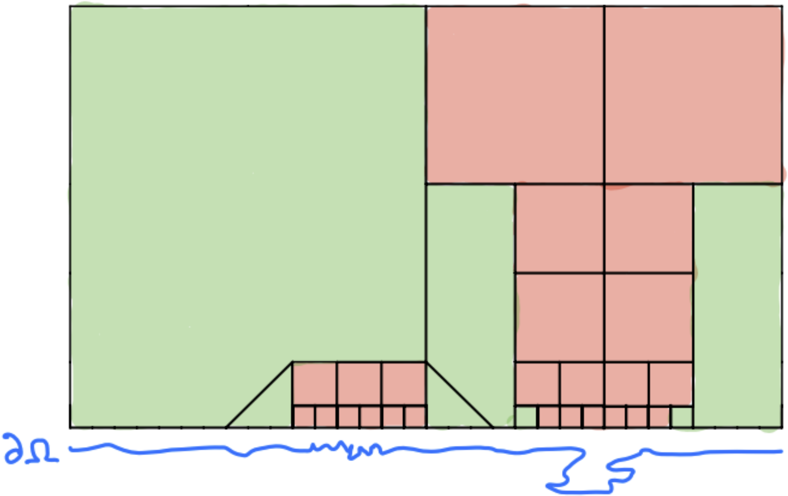
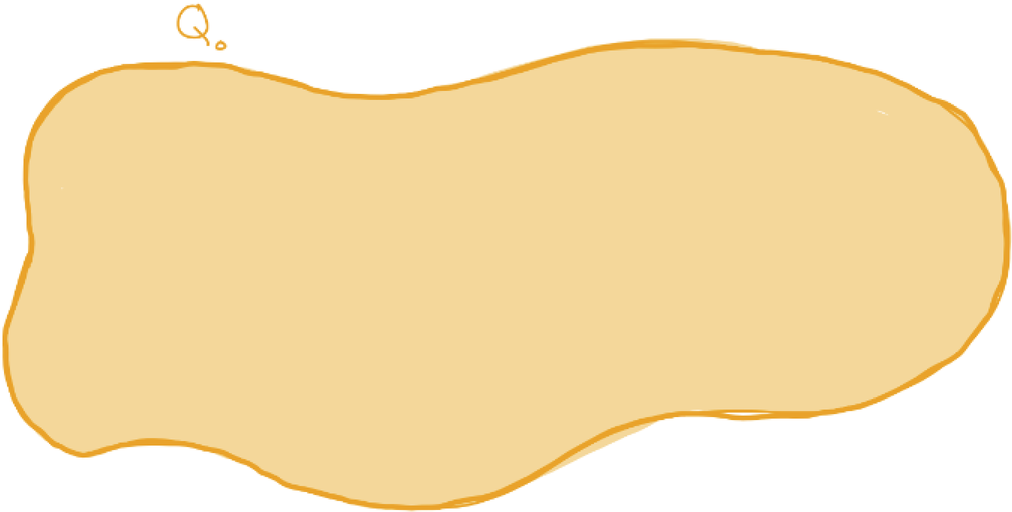
3. Carve up what remains into Lipschitz domains without ruining surface area estimates.



Proof of the Extension Theorem



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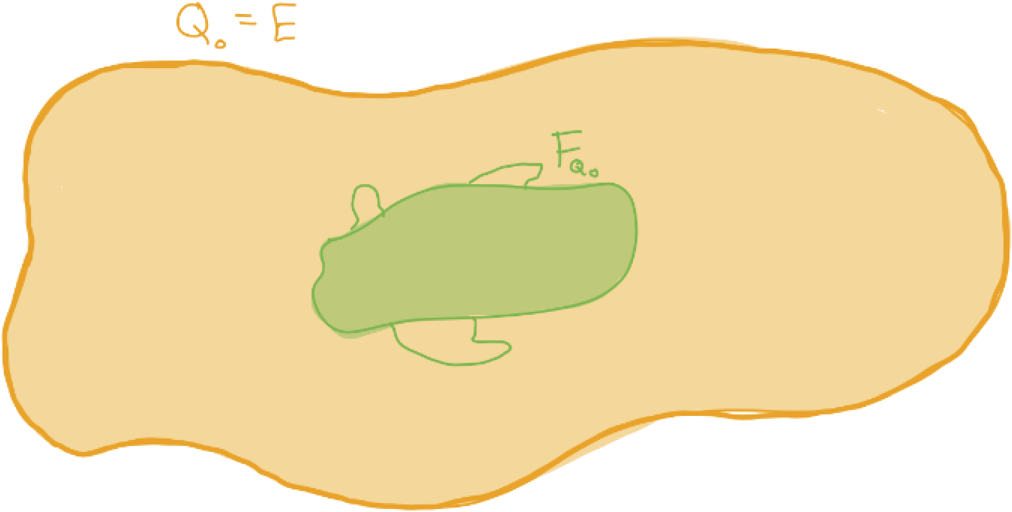

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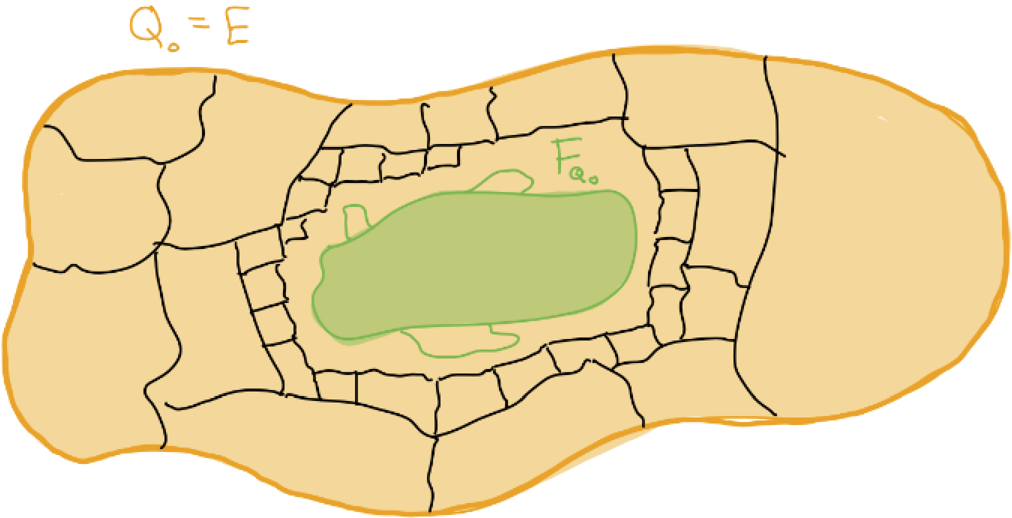

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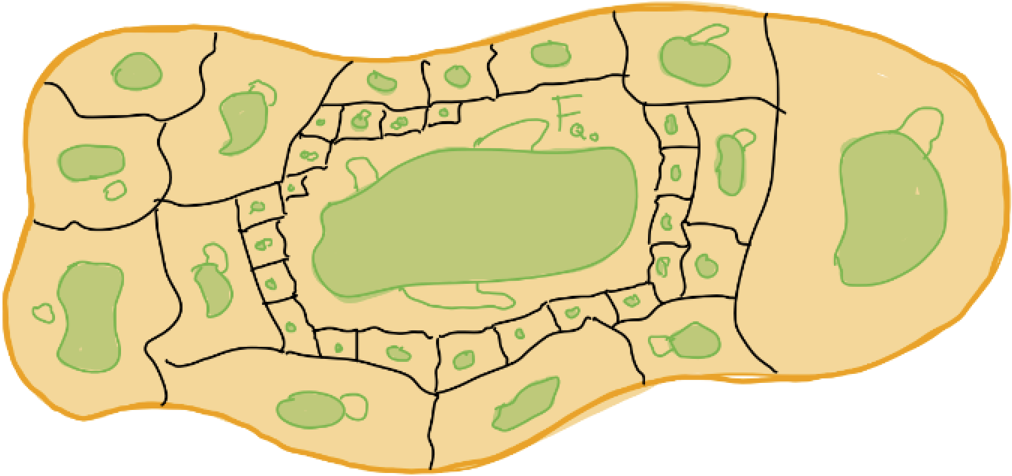
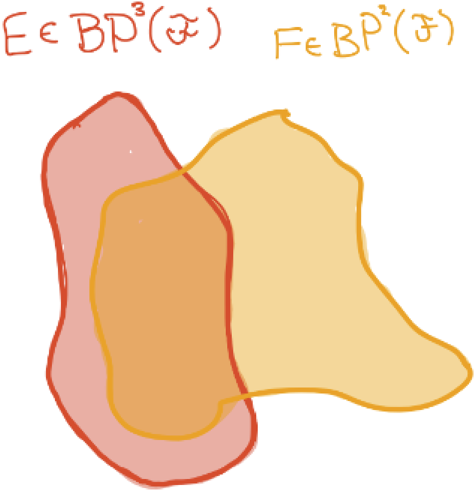


Proof of the Corollary: $E \in \mathrm{BP}^3(\mathscr{F}) \implies E \in \mathrm{BP}^2(\mathscr{F})$



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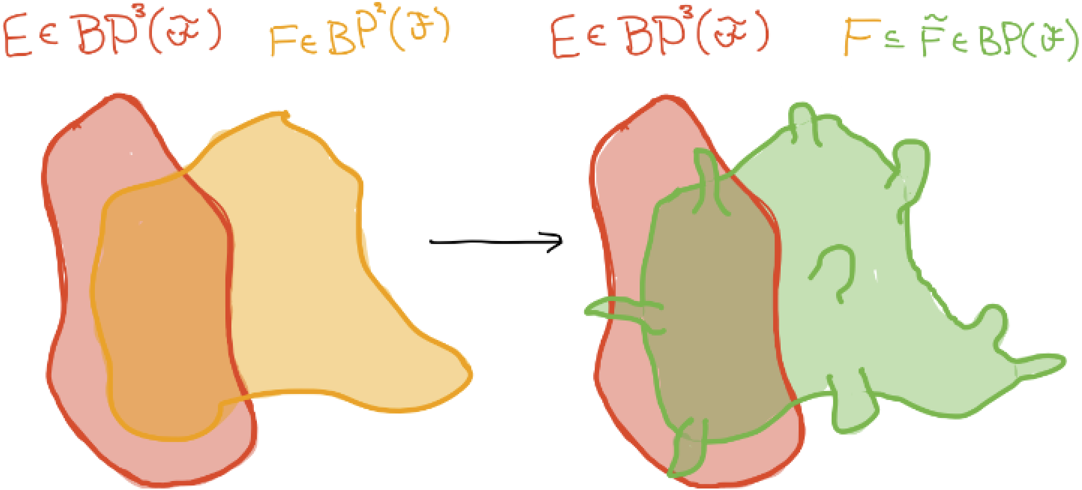
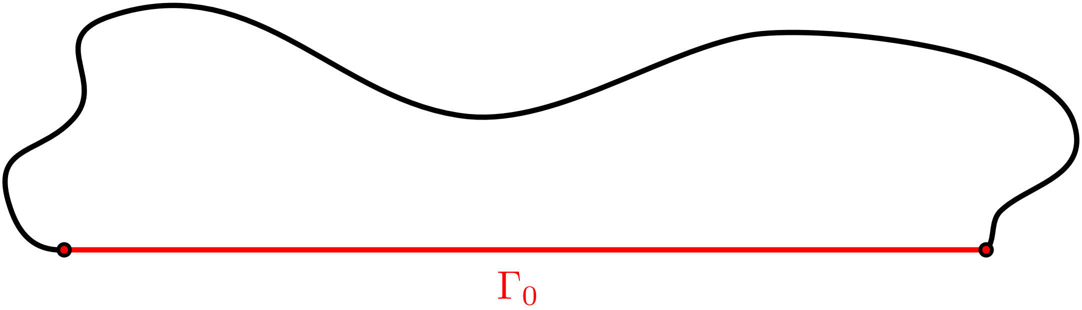


$$\ell(\Gamma) - \mathrm{crd}(\Gamma) = \sum_{i=0}^{\infty} \ell(\Gamma_{i+1}) - \ell(\Gamma_i) \approx \sum_{i=0}^{\infty} \sum_{\substack{Q \in \mathscr{H} \\ \mathrm{rad}(Q) \asymp 2^{-i}}} \beta_{\Gamma}(Q)^2 \, \mathrm{diam}(Q)$$



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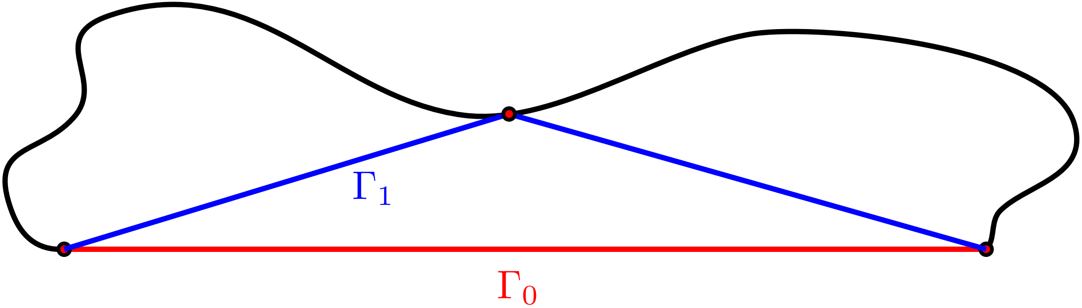

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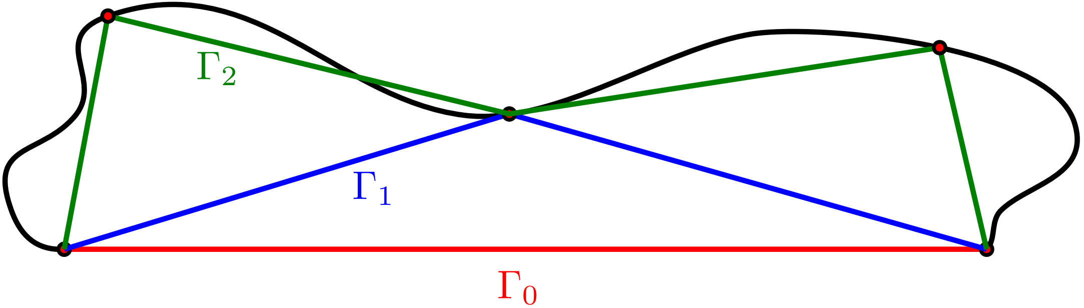

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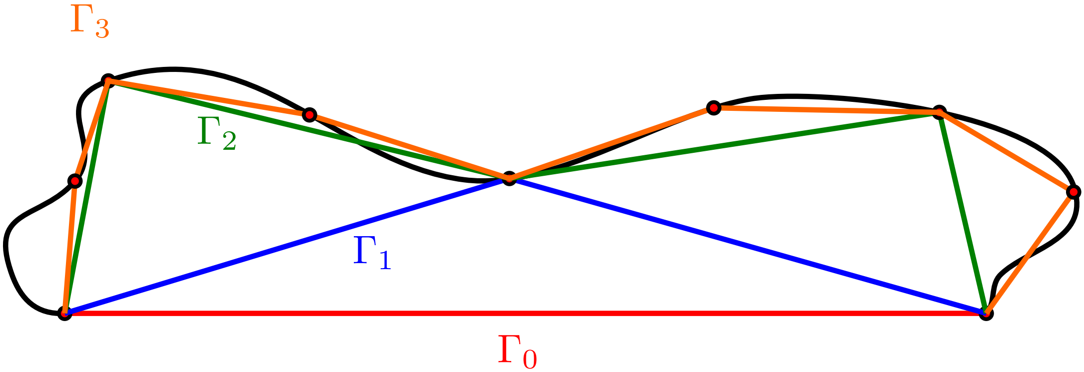

For cubes with "flat arcs", we require Schul's geometric martingales combined with a new tool: Let $\rho(t) = 1 - \gamma_1'(t)$. We introduce an "excess length" measure $d\mu = \gamma_*[\rho dt]$. Notice

$$\mu(\Gamma) = \int_0^{\ell(\Gamma)} (1 - \gamma_1'(t))dt = \ell(\Gamma) - \mathrm{crd}(\Gamma).$$