Uniformly rectifiable metric spaces satisfy the weak constant density condition

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Rectifiability

Definition (n-rectifiable sets)

We say $E \subseteq \mathbb{R}^d$ is n-rectifiable, $n < d$, if $\mathscr{H}^n(E) < \infty$ and there exist a countable collection of Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

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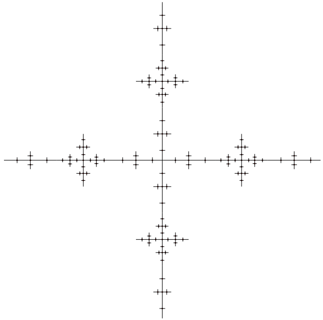

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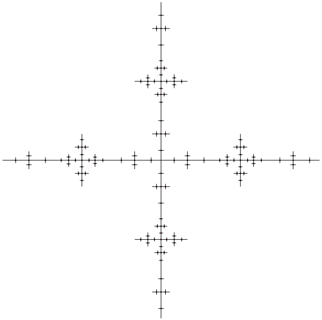
- Give generalization of n-dimensional smooth submanifolds of $\mathbb{R}^d$.
- For $\mathscr{H}^n$ almost every $x \in E$,

$$\lim_{r\to 0} \frac{\mathscr{H}^n(B(x,r)\cap E)}{(2r)^n} = 1.$$



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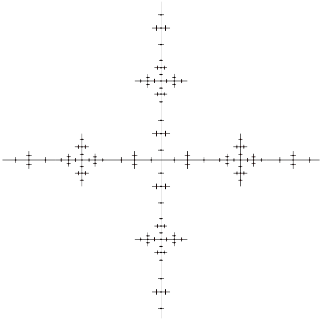
- For $E \subseteq X$, use Lipschitz images of *subsets* of $\mathbb{R}^n$.



The Idea

1. If $E \subseteq \mathbb{R}^d$ is n-rectifiable, then volume locally looks like $\mathscr{L}^n$ on $\mathbb{R}^n$: The density of $\mathscr{H}^n|_E$ approaches 1.
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2. David and Semmes developed a quantitative theory of *uniformly n-rectifiable* subsets of $\mathbb{R}^d$ and proved they satisfy the WCD: they have *nearly constant* density at *most scales and locations*.
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Theorem (K.)

Uniformly n-rectifiable metric spaces satisfy the WCD.

Uniform n-rectifiability

Definition (uniform n-rectifiability)

We say that a set $E \subseteq \mathbb{R}^d$ is *uniformly n-rectifiable* if there exists a constant $C_0 > 0$ such that E is *Ahlfors* (C_0, n)*-regular*, i.e., for all $x \in E$ and $0 < r < \mathrm{diam}(E)$,

$$C_0^{-1} r^n \leq \mathscr{H}^n(B(x,r) \cap E) \leq C_0 r^n, \tag{1}$$

and E has *Big Pieces of Lipschitz images of* $\mathbb{R}^n$ (BPLI), i.e., there exist constants $L, \theta > 0$ such that for all $x \in E$ and $0 < r < \mathrm{diam}(E)$, there exists an L-Lipschitz map $f : B(0,r) \subseteq \mathbb{R}^n \to \mathbb{R}^d$ such that

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David and Semmes gave many equivalent definitions in terms of geometric good/bad decomposition :

$$\begin{aligned} \mathscr{G} &= \{(x,t) \in E \times \mathbb{R}^+ \ : \ B(x,t) \cap E \text{ looks "good"}\}, \\ \mathscr{B} &= \{(x,t) \in E \times \mathbb{R}^+ \ : \ B(x,t) \cap E \text{ looks "bad"}\}, \\ &= E \times \mathbb{R}^+ \setminus \mathscr{G}, \end{aligned}$$

We require that $\mathscr{B}$ be "uniformly small": $\mathscr{B}$ is a *Carleson set*.

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For the WCD,

$$\begin{aligned} \mathscr{G} &= \left\{ (x,t) \in E \times \mathbb{R}^+ \ : \ \frac{\mathscr{H}^n|_E(B(y,r))}{(2r)^n} \approx \text{constant for } B(y,r) \subseteq B(x,t), r \geq \epsilon t \right\}, \\ \mathscr{B} &= E \times \mathbb{R}^+ \setminus \mathscr{G}. \end{aligned}$$

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Carleson Conditions

First, pretend $E \subseteq \mathbb{R}^n$. Families of balls $\approx$ families of cubes via $Q \longleftrightarrow \{(x,t) : x \in Q,\ \frac{\ell(Q)}{2} \leq t \leq \ell(Q)\}$.

Definition (Carleson sets in $\mathbb{R}^n$)

$\mathscr{B} \subseteq \mathcal{D}(\mathbb{R}^n)$ is a *Carleson set* if there exists a constant $C_1 > 0$ such that for all $Q \in \mathcal{D}(\mathbb{R}^n)$,

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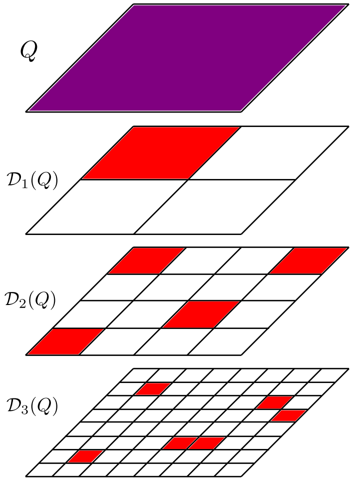
One can associate a "dyadic structure" to E that transfers this logic to general case.



The WCD

Definition (Weak constant density condition)

Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular, let $\epsilon_0 > 0$, and define

$$\mathscr{G}(\epsilon_0) = \left\{ (x,t) \in E \times \mathbb{R}^+ \,\middle|\, \begin{array}{l} \exists c_{(x,t)} > 0 \text{ such that } \forall y \in B(x,t) \cap E,\ 0 < r \leq t, \\ |\mathscr{H}^n(E \cap B(y,r)) - c_{(x,t)} r^n| \leq \epsilon_0 t^n \end{array} \right\},$$

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Theorem

Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular. E is uniformly n-rectifiable if and only if E satisfies the WCD.

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Let $E \subseteq \mathbb{R}^d$ be Ahlfors n-regular. E is uniformly n-rectifiable if and only if E satisfies the WCD.

- Forward implication + reverse implication $n = 1, 2, d-1$ by David and Semmes
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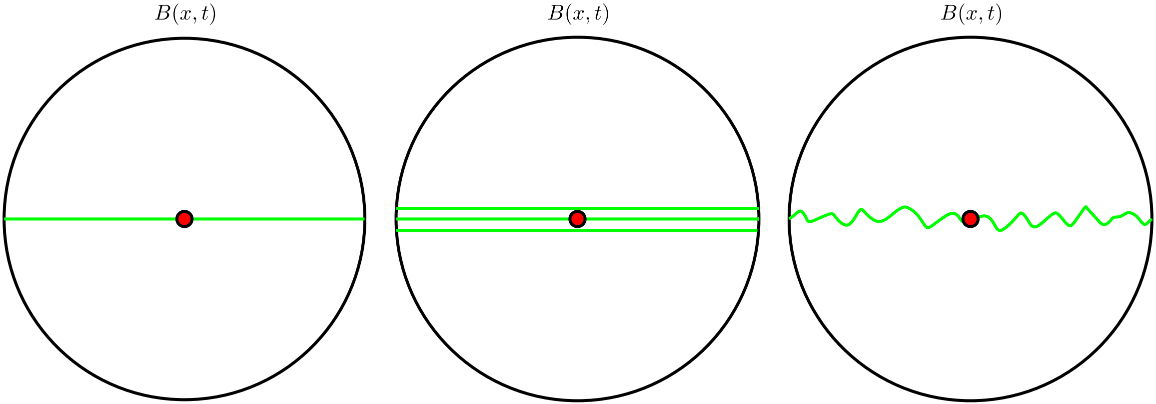
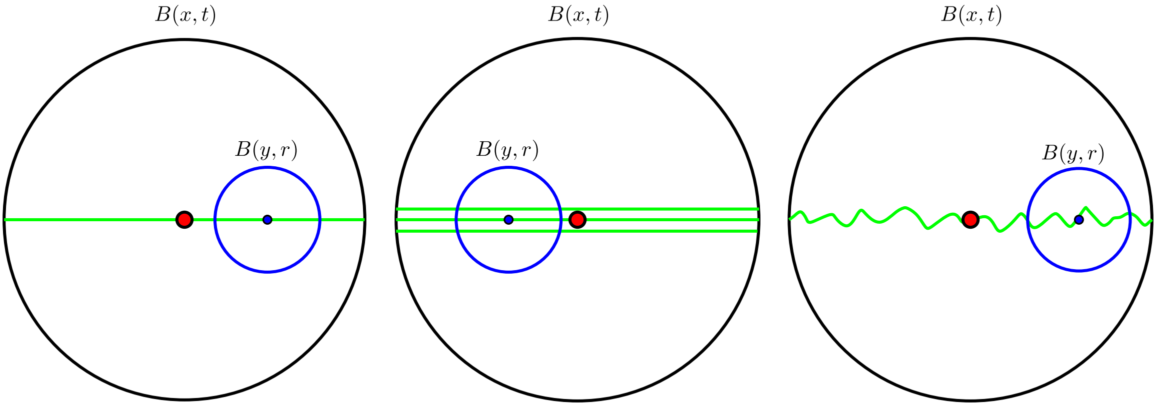
Proof of the WCD: Good balls



Left: Flat now and on all future scales.
Center and right: flat now but not on future scales.

Proof of the WCD: Good balls



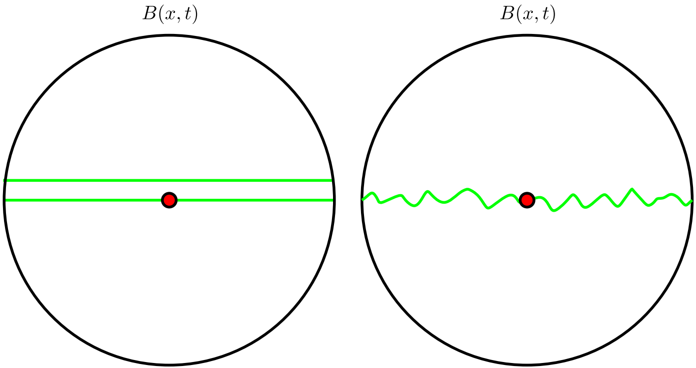
$\mathscr{H}^1(B(y,r) \cap E) = 2r$ $\qquad$ $\mathscr{H}^1(B(y,r) \cap E) \approx 6r$ $\qquad$ $\mathscr{H}^1(B(y,r) \cap E) \approx 2(1+\epsilon)r$

David and Semmes identify bad balls by noticing that their measures are asymmetric and using boundedness of singular integral operators.

Bi-Lip images

Theorem (David and Semmes in $\mathbb{R}^d$; Bate, Hyde, and Schul for metric spaces)

Let X be uniformly n-rectifiable. X has VBPBI, i.e., for all $\epsilon > 0$ there is an $L \geq 1$ such that for each $x \in X$ and $r > 0$ there exists $F \subseteq B(x, r)$, satisfying $\mathscr{H}^n_X(B(x, r) \setminus F) \leq \epsilon r^n$ and an L-bi-Lipschitz map $f : F \to \mathbb{R}^n$.



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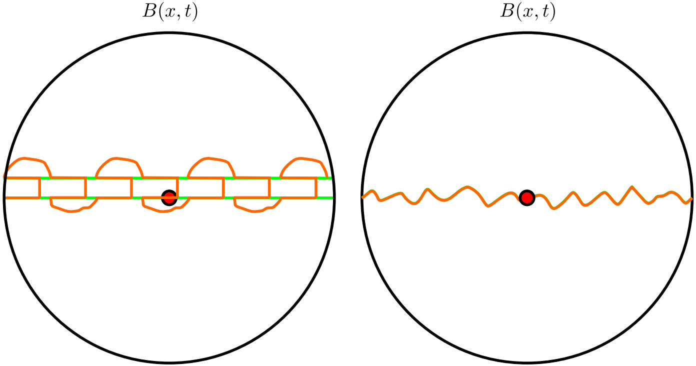
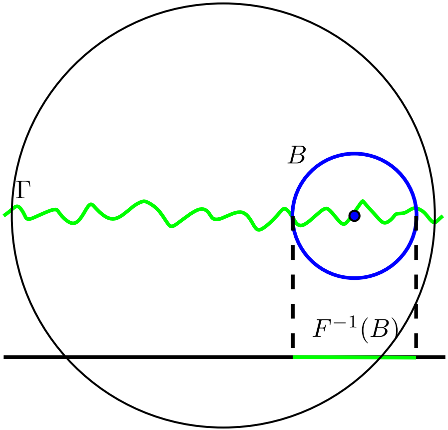


Find symmetry inside bi-Lipschitz images rather than ambient space.

Lipschitz graphs, area formula



Let $F(x) = (x, f(x))$ and $\Gamma = \{F(x) : x \in \mathbb{R}^n\}$. Then,

$$\begin{aligned}\mathscr{H}^n(\Gamma \cap B) &= \int_{F^{-1}(B)} \sqrt{1 + |\nabla f|^2} \\ &\approx \left(\omega_n \fint_{F^{-1}(B)} \sqrt{1 + |\nabla f|^2}\right) \cdot r^n \\ &\approx \left(\omega_n \fint_{F^{-1}(B)} \mathscr{J}(|DF|)\right) \cdot r^n.\end{aligned}$$

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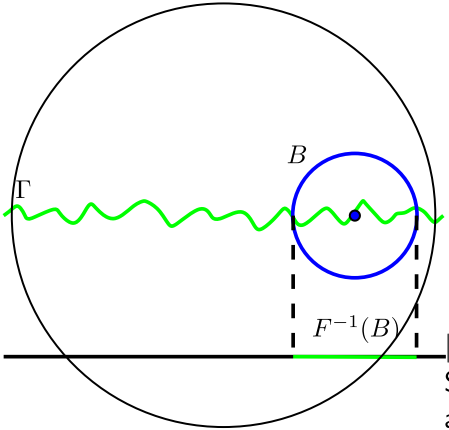

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$$|\mathscr{H}^n(\Gamma \cap B) - c_{x,t} r^n| \approx \left|\left(\omega_n ⨍_{B'} \mathscr{J}(|DF|)\right) - c_{x,t}\right| \cdot r^n.$$

So, it suffices to show $⨍_{B'} \mathscr{J}(|DF|)$ is nearly constant among "largish" B'.

Metric differentiability, Area formula

Definition (Metric derivative, Jacobian)

Let $f : \mathbb{R}^n \to X$ be L-Lipschitz. We say a seminorm on $\mathbb{R}^n$ $|Df|(x)$ is a *metric derivative* of f at x if

$$\lim_{y,z \to x} \frac{d(f(y), f(z)) - |Df|(x)(y-z)|}{|y-x| + |z-x|} = 0.$$

We also have a jacobian $\mathscr{J}_f(x) = \frac{\mathscr{H}^n_{|Df|(x)}(B(0,1))}{\mathscr{L}^n(B(0,1))}$ and area formula.

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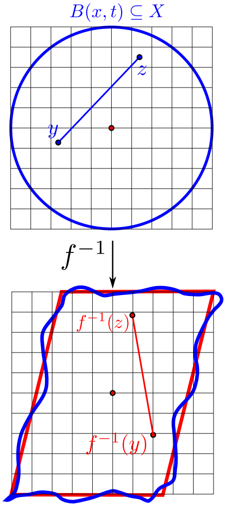
Theorem (Kircheim; Azzam, Schul)

Let $f : \mathbb{R}^n \to X$ be an L-Lipschitz function. f has a metric derivative at $\mathscr{L}^n$ a.e. $x \in \mathbb{R}^n$. For $Q \in \mathcal{D}(\mathbb{R}^n)$, set

$$\mathrm{md}_f(Q) = \inf_{\|\cdot\|} \sup_{x,y\in Q} \frac{1}{\ell(Q)} \left| d(f(x), f(y)) - \|x-y\| \right|.$$

The set $\mathscr{B} = \{Q \in \mathcal{D} : \mathrm{md}_f(Q) > \delta\}$ is a Carleson set for any $\delta > 0$.



Harmonic analysis

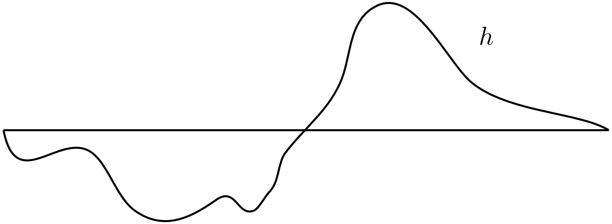
Definition

For $h \in L^2(\mathbb{R}^n)$, $Q \in \mathcal{D}(\mathbb{R}^n)$ define

$$\Delta_Q h(x) = \begin{cases} \fint_P h(z)dz - \fint_Q h(z)dz & \text{if } x \in P \in \mathcal{D}_1(Q), \\ 0 & \text{otherwise} \end{cases}$$

We have $h = \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \Delta_Q h$ orthogonally. Define

$$\Delta_k^h(Q)^2 = \sum_{R \in \mathcal{D}_j(Q),\ j \leq k} \|\Delta_R h\|_2^2$$



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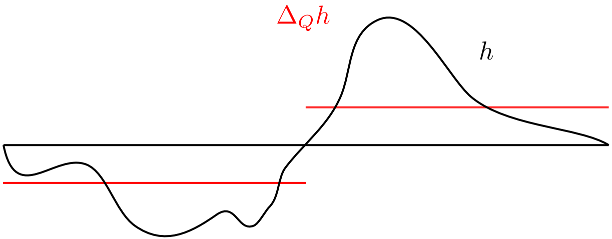

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We have $h = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \Delta_Q h$ orthogonally so $\|h\|_2 = \sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \|\Delta_Q h\|_2^2$. Define

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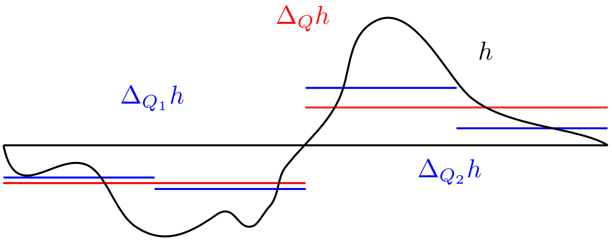

Controlling L^2 mean oscillation

Lemma

There exist $k(M,\epsilon,L) \in \mathbb{N}$ *and* $\delta(M,\epsilon,L) > 0$ *such that the following holds:*
Let $Q \in \mathcal{D}(\mathbb{R}^n)$ *and* $h \in L^2(\mathbb{R}^n)$ *with* $h \geq 0$ *and* $\|h\|_2 \leq M$*. If* $\Delta_k^h(Q)^2 \leq \delta \ell(Q)^n$*, then for any normed ball* L *bi-Lipschitz to* $B(0,\ell(Q))$*, we have*

$$\left| \fint_B h - \fint_Q h \right| \leq \epsilon \left| \fint_Q h \right| .$$

Proof of the lemma

Lemma

$h \in L^2,\ \|h\|_2 \leq M,\ \Delta^h_{k(M,\epsilon,L)}(Q)^2 \leq \delta(M,\epsilon,L)\ell(Q)^n \implies \left|\fint_B h - \fint_Q h\right| \leq \epsilon \left|\fint_Q h\right|$ for all $B \in \mathcal{B}_L(Q)$.

Proof: Suppose the lemma does not hold. After re-scaling, we get a sequence $h_j \in L^2(Q_0),\ Q_0 = [0,1]^n$ with normed balls $B_j \subseteq Q_0$ such that

(i) $\|h_j\|_2 \leq M$,

(ii) $\Delta_j^{h_j}(Q_0) \leq \frac{1}{j}$,

(iii) $\left|\fint_{B_j} h_j - \fint_Q h_j\right| > \epsilon \left|\fint_Q h_j\right|$

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$h \in L^2,\ \|h\|_2 \leq M,\ \Delta^h_{k(M,\epsilon,L)}(Q)^2 \leq \delta(M,\epsilon,L)\ell(Q)^n \implies \left|⨍_B h - ⨍_Q h\right| \leq \epsilon \left|⨍_Q h\right|$ *for all* $B \in \mathcal{B}_L(Q)$.

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By compactness and (i), we can assume $h_j \rightharpoonup h \in L^2(Q_0)$ and $B_j \to B \in \mathcal{B}_L(Q)$. Furthermore, we can use (ii) to show that $\Delta_Q h = 0$ for all $Q \subseteq Q_0$ so that $h = c \in \mathbb{R}$.

Proof of the lemma

Lemma

$h \in L^2,\ \|h\|_2 \leq M,\ \Delta^h_{k(M,\epsilon,L)}(Q)^2 \leq \delta(M,\epsilon,L)\ell(Q)^n \implies \left|\fint_B h - \fint_Q h\right| \leq \epsilon \left|\fint_Q h\right|$ *for all* $B \in \mathcal{B}_L(Q)$.

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$$\lim_j \left|\fint_{B_j} h_j - \fint_{Q_0} h\right| = 0$$

contradicting (iii) for large j.

Controlling L^2 mean oscillation

Lemma

There exist $k(M,\epsilon,L) \in \mathbb{N}$ and $\delta(M,\epsilon,L) > 0$ such that the following holds:
Let $Q \in \mathcal{D}(\mathbb{R}^n)$ and $h \in L^2(\mathbb{R}^n)$ with $h \geq 0$ and $\|h\|_2 \leq M$. If $\Delta_k^h(Q)^2 \leq \delta \ell(Q)^n$, then for any normed ball L bi-Lipschitz to $B(0,\ell(Q))$, we have

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Corollary

Let $f : [0,1]^n \twoheadrightarrow \Sigma$ *be L-bi-Lipschitz. Let*

$$\mathscr{G} = \left\{ Q \in \mathcal{D}(\mathbb{R}^n) : \left| \fint_B \mathscr{J}_f - \fint_Q \mathscr{J}_f \right| \leq \epsilon \left| \fint_Q \mathscr{J}_f \right| \text{ for all } B \in \mathcal{B}_L(Q) \right\} .$$

$\mathscr{B} = \mathcal{D}(\mathbb{R}^n) \setminus \mathscr{G}$ *is Carleson.*

Controlling density oscillation

Let $c_Q = \fint_Q \mathscr{J}_f$. Then for Carleson almost every $Q \in \mathcal{D}$ and all $B \in \mathcal{B}_L(Q)$,

$$\left| \fint_B \mathscr{J}_f - \fint_Q \mathscr{J}_f \right| \leq \epsilon \left| \fint_Q \mathscr{J}_f \right| \iff \left| \int_B \mathscr{J}_f - \mathscr{L}(B) \left(\fint_Q \mathscr{J}_f \right) \right| \leq \epsilon \left(\fint_Q \mathscr{J}_f \right) \mathscr{L}(B)$$
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Assuming $B = B_{\|\cdot\|_Q}(y, r)$, we get $\mathscr{L}(B) = c_{\|\cdot\|_Q} r^n$ so that

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Lemma

Let $f : [0,1]^n \twoheadrightarrow \Sigma$ be L-bi-Lipschitz. Then for Carleson almost every $Q \in \mathcal{D}$ there exists a norm $\|\cdot\|_Q$ achieving $\mathrm{md}_f(Q) \leq \delta$ and

$$\left|\mathscr{H}^n(f(B_{\|\cdot\|_Q}(y, r))) - a_Q r^n\right| \leq \epsilon_0 \ell(Q)^n$$

for all $B_{\|\cdot\|_Q}(y, r) \in \mathcal{B}_L(Q)$ where $a_Q = c_Q c_{\|\cdot\|_Q}$.

Proof the WCD for bi-Lipschitz images

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- WLOG, assume there is $Q_{x,t} \in \mathcal{D}$ satisfying
  (i) $B(x, 3t) \subseteq f(Q_{x,t})$ and $\ell(Q_{x,t}) \asymp_L t$
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- $\mathrm{md}_f(Q_{x,t}) \leq \delta \implies f^{-1}(B(y,r)) \approx B_{\|\cdot\|_{Q_{x,t}}}(f^{-1}(y), r)$. Define $a_{x,t} = a_{Q_{x,t}} = c_{Q_{x,t}} c_{\|\cdot\|_{Q_{x,t}}}$.

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- The lemma implies

$$
\begin{aligned}
&|\mathscr{H}^n(B(y,r)) - a_{x,t} r^n| \\
&\qquad \leq |\mathscr{H}^n(f(f^{-1}(B(y,r)))) - \mathscr{H}^n(f(B_{\|\cdot\|_Q}(f^{-1}(y), r)))| \\
&\qquad\quad + |\mathscr{H}^n(f(B_{\|\cdot\|_Q}(f^{-1}(y), r))) - a_Q r^n| \\
&\qquad \lesssim \epsilon \ell(Q)^n \lesssim \epsilon t^n.
\end{aligned}
$$

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